Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2026) 50: 314 – 331
doi:10.55730/1300-0098.3653
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The Killing magnetic curves in the anti-de Sitter space $\mathbb{H}_1^3$

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Received: 01.09.2025 • **Accepted/Published Online:** 22.12.2025 • **Final Version:** 10.03.2026

Abstract: In this paper, we study space-like, time-like and light-like Killing magnetic curves derived from Killing magnetic vector fields of the anti-de Sitter space $\mathbb{H}_1^3$ by using the half-space model $\mathcal{H}_1^3$ of $\mathbb{H}_1^3$. We find the first integrals of the system of nonlinear differential equations that describe the Killing magnetic curves corresponding to some Killing vector fields of $\mathcal{H}_1^3$, and then we give some particular solutions to obtain space-like, time-like and light-like magnetic curves of $\mathcal{H}_1^3$. Also, we calculate the curvature and torsion of some space-like and time-like Killing magnetic curves, and the torsion of light-like Killing magnetic curves of $\mathcal{H}_1^3$.

Key words: Anti-de Sitter space, magnetic curve, Killing vector, Killing magnetic curve, curvature, torsion

1. Introduction

Let (M_q^n, g) denote an n-dimensional pseudo-Riemannian manifold with index q, and ∇ its Levi-Civita connection. A *magnetic field* is a closed 2-form F on (M_q^n, g) and the *Lorentz force* associated with the magnetic field F on (M_q^n, g) is the skew symmetric $(1,1)$-tensor Φ given by

$$g(\Phi(X), Y) = F(X, Y) \tag{1.1}$$

for any vector fields X, Y tangent to M_q^n. A magnetic trajectory of a magnetic field F is a curve γ in M_q^n, also called magnetic curve, if its velocity vector γ' satisfies the *Lorentz equation*

$$\nabla_{\gamma'}\gamma' = \Phi(\gamma'). \tag{1.2}$$

Magnetic curves obviously generalize geodesics. When the magnetic field F is trivial, $F = 0$, the Lorentz force Φ is zero, and thus the Lorentz equation (1.2) reduces to the equation of geodesics. Therefore, in the view of dynamical systems, a geodesic corresponds to a trajectory of a particle without an influence of a magnetic field. As the Lorentz force is skew symmetric, it can be seen easily that a magnetic curve has a constant speed, that is, $||\gamma'|| =$ constant. A magnetic curve parameterized by arc length parameter s is called a *normal* magnetic curve.

A wide literature is devoted to the study of magnetic flows and curves, motivated by their natural occurrence in several areas of mathematical physics. Their study sits at the intersection of differential geometry, dynamical systems, and mathematical physics. Furthermore, magnetic curves have been shown to correspond

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2010 *AMS Mathematics Subject Classification:* 53B25, 53B30

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to Kirchhoff elastic rods, establishing a link between elasticity theory and the Hall effect, and they also arise as critical points of the Landau-Hall functional. For these reasons, the study of magnetic curves in Riemannian and pseudo-Riemannian manifolds has become an active research area in both mathematics and mathematical physics [2, 4–6, 8, 13, 15]. Anti-de Sitter space $\mathbb{H}^3_1$ is of particular interest not only as a Lorentzian counterpart to hyperbolic space but also due to its role as a maximally symmetric spacetime of constant negative curvature. A complete classification of its Killing magnetic curves enriches the catalogue of exactly solvable models in Lorentzian geometry and offers a foundation for further exploration in geometric mechanics and related fields.

On a 3-dimensional (semi-)Riemannian manifold, 2-forms can be identified with vector fields. In this framework, magnetic fields correspond to divergence-free vector fields. Since Killing vector fields are divergence-free, they define an important class of magnetic fields known as *Killing magnetic fields*. Over the past two decades, Killing magnetic curves corresponding to Killing magnetic fields have been systematically investigated in various classical ambient spaces, including the Euclidean space $\mathbb{E}^3$ [2, 12, 16], the Minkowski space $\mathbb{E}^3_1$ [3], the hyperbolic space $\mathbb{H}^3$ [6, 9], the sphere $\mathbb{S}^3$ [1], Sol space [5], the product spaces $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$ [11, 13], and in the anti-de Sitter space $\mathbb{H}^3_1$ [7]. However, despite its physical and geometric importance, a comprehensive classification of Killing magnetic curves in $\mathbb{H}^3_1$ has remained incomplete. Previous work by Calvaruso–Munteanu [7] studied magnetic curves associated with the hyperbolic Hopf vector field via a paraquaternionic approach, obtaining a full description only for light-like curves. A general classification for all six independent Killing vector fields of $\mathbb{H}^3_1$ has not been undertaken.

In the Minkowski 3-space $\mathbb{E}^3_1$, Drută-Romaniuc and Munteanu [3] classified all magnetic curves generated by the Killing vector field $V = a\partial_x + b\partial_y + c\partial_z$. They proved that, depending on the causal character of V, the magnetic curves are explicit helices: timelike V yields cylindrical helices, spacelike V gives hyperbolic helices, and lightlike V produces cubic-type curves. In all cases the curvature and torsion are constant whenever defined, confirming that Killing magnetic curves in $\mathbb{E}^3_1$ are helices—the Lorentzian analogues of the Euclidean helical trajectories found in [2].

In [17], Seppi and Trebeschi developed a half-space model $\mathcal{H}^{p,q}$ for the $(p+q)$-dimensional pseudo-hyperbolic space $\mathbb{H}^{p,q}$ of signature (p,q) with $q \geq 1$. They showed that there exists an isometric embedding of this space into $\mathbb{H}^{p,q}$, and its image is the complement of a totally geodesic degenerate hyperplane in $\mathbb{H}^{p,q}$. They also described full isometry group of $\mathcal{H}^{p,q}$. In this work, we are motivated by the fact that the anti-de Sitter space $\mathcal{H}^3_1$ admits a richer variety of magnetic curves than the hyperbolic space $\mathbb{H}^3$, primarily due to the causal character of the principal tangent vector on $\mathcal{H}^3_1$.

We first derive a system of nonlinear differential equations that describe the Killing magnetic curves corresponding to each Killing vector field of $\mathcal{H}^3_1$. We then determine the first integrals of this system for selected Killing vector fields. Also, we give some particular solutions to obtain space-like, time-like and light-like Killing magnetic curves of $\mathcal{H}^3_1$. Moreover, we calculate the curvature and torsion of some space-like and time-like Killing magnetic curves, and the torsion of light-like Killing magnetic curves of $\mathcal{H}^3_1$.

2. Preliminaries

Let (M^3_1, g) be a $3-$dimensional Lorentzian manifold. For the basics of Lorentizian manifolds we refer to [10, 14]. A nonzero vector $v \in T_pM^3_1$ at a point $p \in M^3_1$ is said to be space-like, time-like, or light-like (null) if $g(v,v) > 0$, $g(v,v) < 0$, or $g(v,v) = 0$, respectively. The norm of a vector v is defined by $\|v\| = \sqrt{|g(v,v)|}$.

A curve $\gamma : J \to M_1^3$ from an open interval $J \subset \mathbb{R}$ to a 3-dimensional pseudo-Riemannian manifold M_1^3 is called *space-like* (resp., *time-like* or *light-like*) if its tangent vector $\gamma'(s)$ on J is space-like (resp., time-like or light-like).

A nonlight-like curve $\gamma(s)$ in M_1^3 is said to be a unit speed curve if $g(\gamma'(s), \gamma'(s)) = \pm 1$, and s is called the *arc-length* parameter. A unit speed curve $\gamma(s)$ in M_1^3 is called a geodesic if $\nabla_{\gamma'(s)}\gamma'(s) = 0$ along $\gamma(s)$.

If $\gamma(s)$ is a nongeodesic light-like curve in M_1^3, then $\nabla_{\gamma'(s)}\gamma'(s)$ is space-like along γ. It can be reparameterized such that $g(\nabla_{\gamma'(s)}\gamma'(s), \nabla_{\gamma'(s)}\gamma'(s)) = 1$. This parameter s is called a pseudo arc-length parameter, and the curve is referred to as a unit speed curve.

A unit speed space-like or time-like curve $\gamma(s)$ in M_1^3 is called a Frenet curve if $g(\nabla_{\gamma'(s)}\gamma'(s), \nabla_{\gamma'(s)}\gamma'(s))$ $\neq 0$ for all $s \in J$. A Frenet curve admits an orthonormal frame field $\{T = \gamma'(s), N, B\}$ along γ satisfying the Frenet equations

$$\nabla_T T = \varepsilon_N \kappa N, \tag{2.1}$$

$$\nabla_T N = -\varepsilon_T \kappa T + \varepsilon_B \tau B, \tag{2.2}$$

$$\nabla_T B = -\varepsilon_N \tau N, \tag{2.3}$$

where $\kappa = \kappa(s) = ||\nabla_T T||$ and $\tau = \tau(s) = -g(\nabla_T B, N)$ are the curvature and torsion functions of γ, respectively; N is the principal normal vector, B is the binormal vector, and $\varepsilon_T = g(T,T) = \pm 1, \varepsilon_N = g(N,N) = \pm 1$ and $\varepsilon_B = g(B,B) = \pm 1$ denote the causal characters of T, N and B, respectively, such that $\varepsilon_T \varepsilon_N \varepsilon_B = -1$.

Let $\gamma(s)$ be a nongeodesic unit speed light-like curve in M_1^3 for all $s \in J$ with a pseudo-orthonormal Frenet frame (also known as Cartan frame) $\{T, N, B\}$, i.e., $T = \gamma', N, B$ are the vector fields tangent to M_1^3 along γ satisfying the conditions:

$$g(T,T) = g(B,B) = g(T,N) = g(B,N) = 0,\ g(N,N) = g(T,B) = 1$$

and

$$\nabla_T T = N, \tag{2.4}$$

$$\nabla_T N = \tau T - B, \tag{2.5}$$

$$\nabla_T B = -\tau N, \tag{2.6}$$

where $\tau = \tau(s) = g(\nabla_T N, B)$ is the torsion function of γ.

We consider the upper half-space model $\mathcal{H}_1^3 = \mathcal{H}^{2,1}$ for the anti-de Sitter space $\mathbb{H}_1^3$ which is defined as the open half-space $\{(x,y,z) \in \mathbb{R}_1^3 | \, z > 0\}$ equipped with the Lorentzian metric $g = \dfrac{dx^2 - dy^2 + dz^2}{z^2}$, of constant curvature -1. The boundary of $\mathcal{H}_1^3$, denoted by $\partial\mathcal{H}_1^3$, is the Minkowski 2-plane $\{z = 0\}$.

Let ∇ denote the connection of $\mathcal{H}_1^3$. Let $\partial_x = \frac{\partial}{\partial x}, \partial_y = \frac{\partial}{\partial y}, \partial_z = \frac{\partial}{\partial z}$ denote coordinate vector fields on $\mathcal{H}_1^3$. Then, the vector fields $E_1 = z\partial_x, E_2 = z\partial_y, E_3 = z\partial_z$ with $g(E_1, E_1) = g(E_3, E_3) = 1,\ g(E_2, E_2) = -1$ form a pseudo-orthnormal frame on $\mathcal{H}_1^3$. In this frame, nonzero covariant derivatives of $\mathcal{H}_1^3$ are obtained as follows

$$\nabla_{E_1} E_1 = E_3,\ \nabla_{E_1} E_3 = -E_1,\ \nabla_{E_2} E_2 = -E_3,\ \nabla_{E_2} E_3 = -E_2.$$

Let $\gamma(s) = (x(s), y(s), z(s))$ be a unit speed curve in $\mathcal{H}_1^3$, that is, it holds

$$x'^2 - y'^2 + z'^2 = \varepsilon_T z^2. \tag{2.7}$$

The curve γ is space-like when $\varepsilon_T = 1$, and time-like when $\varepsilon_T = -1$. A curve $\gamma(s) = (x(s), y(s), z(s))$ is light-like if (2.7) holds for $\varepsilon_T = 0$, where s is an arbitrary parameter for $\gamma(s)$.

Now, by a direct computation we obtain that

$$\nabla_{\gamma'}\gamma' = \Big(x'' - \frac{2x'z'}{z}\Big)\partial_x + \Big(y'' - \frac{2y'z'}{z}\Big)\partial_y + \Big(z'' + \frac{x'^2 - y'^2 - z'^2}{z}\Big)\partial_z \tag{2.8}$$

form which, following [17], one can determine all geodesics of $\mathcal{H}_1^3$.

On the anti-de Sitter space $\mathcal{H}_1^3$, a magnetic field is defined as a closed 2-form F which may be identified with a corresponding vector field using the Hodge star operator and the volume form Ω_3 of $\mathcal{H}_1^3$, that is, for a vector field $V \in TM$

$$F_V(X, Y) = \Omega_3(V, X, Y),\ \forall X, Y \in TM. \tag{2.9}$$

Also, the cross product of two vector fields $X, Y \in TM$ is defined as follows

$$\tilde{g}(X \wedge Y, Z) = \Omega_3(X, Y, Z),\ \forall Z \in TM. \tag{2.10}$$

Since the volume element of $\mathcal{H}_1^3$ is $\Omega_3 = \frac{1}{z^3} dx \wedge dy \wedge dz$, for the orthogonal fields $\{\partial_x,\ \partial_y,\ \partial_z\}$, we define the cross product of these vectors by using (2.10) as

$$\partial_x \wedge \partial_y := \frac{1}{z}\partial_z, \quad \partial_x \wedge \partial_z := \frac{1}{z}\partial_y, \quad \partial_y \wedge \partial_z := \frac{1}{z}\partial_x. \tag{2.11}$$

Using (1.1), (2.9) and (2.10), we may express the Lorentz force corresponding to F_V as $\Phi_V(X) = V \wedge X$ from which and (1.2), a curve $\gamma(s)$ is a magnetic curve in $\mathcal{H}_1^3$ if $\gamma(s)$ satisfies

$$\nabla_{\gamma'}\gamma' = V \wedge \gamma'. \tag{2.12}$$

A vector field V on a pseudo-Riemannian manifold M_q^n is a Killing vector field if its local flows are isometries, and Killing vector fields are characterized as vector fields satisfying the Killing equation:

$$g(\nabla_X V, Y) + g(X, \nabla_Y V) = 0,\ \forall X, Y \in TM_q^n.$$

Then, by a straightforward computation, we obtain the Killing vector fields of $\mathcal{H}_1^3$ as follows

$$V_1 = \partial_x, \quad V_2 = \partial_y, \quad V_3 = x\partial_x + y\partial_y + z\partial_z, \quad V_4 = y\partial_x + x\partial_y,$$

$$V_5 = \frac{1}{2}(x^2 + y^2 - z^2)\partial_x + xy\partial_y + xz\partial_z, \quad V_6 = xy\partial_x + \frac{1}{2}(x^2 + y^2 + z^2)\partial_y + yz\partial_z.$$

3. The Killing magnetic curves in $\mathcal{H}_1^3$

Here, we investigate time-like, space-like and light-like Killing magnetic curves corresponding to the magnetic fields defined by Killing vector fields, and we calculate curvature and torsion of some Killing magnetic curves.

3.1. The Killing magnetic curves in $\mathcal{H}_1^3$ corresponding to V_1

Theorem 3.1 *A smooth space-like or time-like unit speed curve $\gamma(s)$ in $\mathcal{H}_1^3$ is a normal Killing magnetic curve corresponding to the Killing vector field $V_1 = \partial_x$ if the curve $\gamma(s)$ is one of the followings:*

1) the space-like Killing magnetic line $\gamma(s) = (\pm\sqrt{2}s, s, 1)$ or

2) the time-like Killing magnetic line $\gamma(s) = (0, -s, 1)$ or

3) the component functions $x(s), y(s), z(s)$ of $\gamma(s)$ satisfy the following equations:

$$x'(s) = v_1 z^2(s), \tag{3.1}$$

$$y'(s) = (v_2 + 1/2)z^2(s) - 1/2, \tag{3.2}$$

$$z'(s) = \pm\sqrt{[(v_2+1/2)^2 - v_1^2]z^4(s) + (\varepsilon_T - v_2 - 1/2)z^2(s) + 1/4} \tag{3.3}$$

with the initial conditions $x(0) = y(0) = 0, z(0) = 1, x'(0) = v_1, y'(0) = v_2,$ and $z'(0) = v_3$ such that $v_1^2 - v_2^2 + v_3^2 = \varepsilon_T$.

In particular, some solutions of these differential equations yield the followings:

A) Space-like Killing magnetic curves:

A-1) $\gamma(s) = \Big(\frac{2}{3}(\frac{s}{2}+1)^3 - \frac{2}{3}, \frac{2}{3}(\frac{s}{2}+1)^3 - \frac{s}{2} - \frac{2}{3}, \frac{s}{2}+1\Big)$, *where* $s > -2$,

A-2) $\gamma(s) = \Big(-\frac{3s}{8} + \lambda_1(s), -\frac{7s}{8} + \lambda_1(s), \sqrt{2}\sinh\frac{s}{2} + \cosh\frac{s}{2}\Big)$,

where $\lambda_1(s) = -\frac{3\sqrt{2}}{4} + \frac{3\sqrt{2}}{4}\cosh s + \frac{9}{8}\sinh s$ *and* $s > -2\tanh^{-1}(\frac{1}{\sqrt{2}})$,

A-3) $\gamma(s) = \Big(\frac{5s}{8} + \frac{5}{8}\sin s, \frac{s}{8} + \frac{5}{8}\sin s, \cos\frac{s}{2}\Big)$, *where* $-\pi < s < \pi$,

A-4) $\gamma(s) = \Big(0, -1 - s + \tan(\frac{s}{2} + \frac{\pi}{4}), \tan(\frac{s}{2} + \frac{\pi}{4})\Big)$, *where* $-\frac{\pi}{2} < s < \frac{\pi}{2}$.

B) Time-like Killing magnetic curves:

B-1) $\gamma(s) = \Big(\frac{2}{3}(\frac{s}{2}+1)^3 - \frac{2}{3}, \frac{2}{3} - \frac{s}{2} - \frac{2}{3}(\frac{s}{2}+1)^3, \frac{s}{2}+1\Big)$, *where* $s > -2$,

B-2) $\gamma(s) = \Big(\frac{5s}{8} + \lambda_2(s), \frac{s}{8} + \lambda_2(s), \sqrt{2}\sinh\frac{s}{2} + \cosh\frac{s}{2}\Big)$,

where $\lambda_2(s) = \frac{5\sqrt{2}}{4} - \frac{5\sqrt{2}}{4}\cosh s - \frac{15}{8}\sinh s$ *and* $s > -2\tanh^{-1}(\frac{1}{\sqrt{2}})$,

B-3) $\gamma(s) = \Big(-\frac{3s}{8} - \frac{3}{8}\sin s, -\frac{7s}{8} - \frac{3}{8}\sin s, \cos\frac{s}{2}\Big)$, *where* $-\pi < s < \pi$,

B-4) $\gamma(s) = \Big(-2\sqrt{2} - \sqrt{2}s + 2\sqrt{2}\tan(\frac{s}{2} + \frac{\pi}{4}), 3 + s - 3\tan(\frac{s}{2} + \frac{\pi}{4}), \tan(\frac{s}{2} + \frac{\pi}{4})\Big)$,

where $-\frac{\pi}{2} < s < \frac{\pi}{2}$.

Proof Let $\gamma(s) = (x(s), y(s), z(s))$ be a unit speed space-like or time-like curve in $\mathcal{H}_1^3$. Then, using (2.11) and (2.12), $\gamma(s)$ is a normal Killing magnetic curve corresponding to the Killing vector field $V_1 = \partial_x$ if $\gamma(s)$ satisfies

$$\nabla_{\gamma'}\gamma' = \partial_x \wedge \Big(x'(s)\partial_x + y'(s)\partial_y + z'(s)\partial_z\Big) = \frac{z'}{z}\partial_y + \frac{y'}{z}\partial_z \tag{3.4}$$

from which and (2.8), we obtain the following system of nonlinear differential equations:

$$zx'' = 2x'z', \tag{3.5}$$

$$zy'' = 2y'z' + z', \tag{3.6}$$

$$zz'' = {y'}^2 + {z'}^2 - {x'}^2 + y'. \tag{3.7}$$

We solve this system of differential equations together with the initial conditions $x(0) = y(0) = 0, z(0) = 1, x'(0) = v_1, y'(0) = v_2$, and $z'(0) = v_3$ such that $v_1^2 - v_2^2 + v_3^2 = \varepsilon_T$.

Let $z = 1$. Then, we have $z' = 0$, that is, $v_3 = 0$, and $v_1^2 - v_2^2 = \varepsilon_T$. Also, from (2.7) and (3.7) we get $y' = \varepsilon_T$, that is, $y(s) = \varepsilon_T s$ with $v_2 = \varepsilon_T$, and from (2.7) we have $x(s) = \pm\sqrt{\varepsilon_T + 1}s$ with $v_1 = \pm\sqrt{\varepsilon_T + 1}$. Therefore, we obtain 1) when $\varepsilon_T = 1$, and 2) when $\varepsilon_T = -1$ in Theorem 3.1.

We suppose that $z(s)$ is a nonconstant function on some open interval containing 0. From the first integrals of (3.5) and (3.6), we obtain (3.1) and (3.2), respectively. By using (2.7), (3.5) and (3.6), in the derivative of (2.7) with respect to s, we can obtain (3.7), that is, a solution of (2.7), (3.5) and (3.6) is a solution of (3.7). Therefore, from (2.7), (3.1) and (3.2), we obtain (3.3).

Now, we consider some particular solutions of (3.1), (3.2) and (3.3) for particular choices of v_1, v_2, v_3, and ε_T.

Case 1. $v_1 = 1$, $v_2 = \varepsilon_T - 1/2$ and $v_3 = 1/2$. Then, we have $z'(s) = \pm 1/2$ from (3.3). We only consider " + " case in z'. Thus, we obtain $z(s) = 1 + \frac{s}{2}$ with $s > -2$, and from (3.1) and (3.2) we have $x(s) = \dfrac{2}{3}[(\dfrac{s}{2} + 1)^3 - 1]$ and $y(s) = \dfrac{2\varepsilon_T}{3}[(\dfrac{s}{2} + 1)^3 - 1] - \dfrac{s}{2}$. Therefore, for $\varepsilon_T = 1$ and $\varepsilon_T = -1$ we obtain A-1) and B-1) in Theorem 3.1, respectively.

Case 2. $v_1 = \varepsilon_T - \frac{1}{4}$, $v_2 = \varepsilon_T - \frac{3}{4}$, and $v_3 = \frac{1}{\sqrt{2}}$. Then, from (3.3) we have $z' = \frac{1}{2}\sqrt{1 + z^2}$ that yields

$$s = \int_1^z \frac{2du}{\sqrt{1 + u^2}} = 2(\sinh^{-1} z - \sinh^{-1}(1)).$$

Hence, we get

$$z(s) = \sqrt{2}\sinh(\frac{s}{2}) + \cosh(\frac{s}{2}) \quad \text{and} \quad z^2(s) = 1 + 3\sinh^2(\frac{s}{2}) + \sqrt{2}\sinh s \tag{3.8}$$

with $s > -2\tanh^{-1}\left(\frac{1}{\sqrt{2}}\right)$. By (3.1), we have $x'(s) = (\varepsilon_T - \frac{1}{4})\Big(1 + 3\sinh^2(\frac{s}{2}) + \sqrt{2}\sinh s\Big)$ which gives

$$x(s) = (\varepsilon_T - \frac{1}{4})\Big(- \frac{s}{2} + \sqrt{2}\cosh s + \frac{3}{2}\sinh s\Big) - \sqrt{2}(\varepsilon_T - \frac{1}{4}).$$

Similarly, from (3.2) we obtain that

$$y(s) = (\varepsilon_T - \frac{1}{4})\Big(\sqrt{2}\cosh s + \frac{3}{2}\sinh s - \sqrt{2}\Big) - \frac{(4\varepsilon_T + 3)s}{8}.$$

Thus, for $\varepsilon_T = 1$ and $\varepsilon_T = -1$ we obtain A-2) and B-2) in Theorem 3.1, respectively.

Case 3. $v_1 = \varepsilon_T + \frac{1}{4}$, $v_2 = \varepsilon_T - \frac{1}{4}$, and $v_3 = 0$. Then, from (3.3) we have $z' = \frac{1}{2}\sqrt{1 - z^2}$ which produces $z(s) = \cos(\frac{s}{2})$ for $-\pi < s < \pi$. From (3.1) and (3.2), we obtain that

$$x(s) = \frac{4\varepsilon_T + 1}{8}s + \frac{4\varepsilon_T + 1}{8}\sin s \quad \text{and} \quad y(s) = \frac{4\varepsilon_T - 3}{8}s + \frac{4\varepsilon_T + 1}{8}\sin s.$$

Therefore, for $\varepsilon_T = 1$ and $\varepsilon_T = -1$, we have A-3) and B-3) in Theorem 3.1, respectively.

Case 4. $v_1 = \sqrt{1-\varepsilon_T}$, $v_2 = \varepsilon_T - 1$, and $v_3 = 1$. Then, we have $z' = \frac{1}{2}(1+z^2)$ from (3.3). Integrating this with the initial conditions yields $z(s) = \tan(\frac{s}{2} + \frac{\pi}{4})$, where $-\pi/2 < s < \pi/2$. Hence, we have from (3.1) and (3.2) that

$$x(s) = 2\sqrt{1-\varepsilon_T}\tan(\frac{s}{2} + \frac{\pi}{4}) - \sqrt{1-\varepsilon_T}s - 2\sqrt{1-\varepsilon_T}$$

and

$$y(s) = (2\varepsilon_T - 1)\tan(\frac{s}{2} + \frac{\pi}{4}) - \varepsilon_T s - 2\varepsilon_T + 1.$$

Thus, for $\varepsilon_T = 1$ and $\varepsilon_T = -1$, we obtain A-4) and B-4) in Theorem 3.1, respectively. □

Corollary 3.2 *Let $\gamma(s) = (x(s), y(s), z(s))$ be a smooth nongeodesic space-like or time-like, unit speed Killing magnetic curve corresponding to the Killing vector field $V_1 = \partial_x$ in $\mathcal{H}_1^3$ with the initial conditions $x(0) = y(0) = 0, z(0) = 1, x'(0) = v_1, y'(0) = v_2$, and $z'(0) = v_3$ such that $v_1^2 - v_2^2 + v_3^2 = \varepsilon_T$. Then, the curvature κ and the torsion τ of $\gamma(s)$ are expressed in terms of z as follows*

$$\kappa(s) = \sqrt{\varepsilon_N(v_1^2 - \frac{\varepsilon_T}{z^2(s)})} \quad \text{and} \quad \tau(s) = \frac{\varepsilon_N v_1\Big[\Big(\varepsilon_T(2v_2+1) - 2v_1^2\Big)z^2(s) + \varepsilon_T\Big]}{2(v_1^2 z^2(s) - \varepsilon_T)},$$

where ε_T and ε_N are the signatures of the tangent vector T and the principal normal vector N, respectively.

Proof Let $\gamma(s)$ be a curve stated in the hypothesis. Then, by using (2.7) and (3.4) we can have

$$\kappa = \|\nabla_T T\| = \|\frac{z'}{z}\partial_y + \frac{y'}{z}\partial_z\| = \sqrt{\varepsilon_N\Big(\frac{y'^2 - z'^2}{z^4}\Big)} = \sqrt{\varepsilon_N\Big(\frac{x'^2 - \varepsilon_T z^2}{z^4}\Big)}.$$

As γ is a Killing magnetic curve, by using (3.1) we obtain that

$$\kappa = \sqrt{\varepsilon_N\Big(\frac{v_1^2 z^4 - \varepsilon_T z^2}{z^4}\Big)} = \sqrt{\varepsilon_N\Big(v_1^2 - \frac{\varepsilon_T}{z^2}\Big)} \tag{3.9}$$

that gives the desired result.

Now, form (2.1), the principal normal vector of a nongeodesic space-like or time-like curve γ is

$$N = \frac{\varepsilon_N}{\kappa}\nabla_T T = \frac{\varepsilon_N}{\kappa}(\frac{z'}{z}\partial_y + \frac{y'}{z}\partial_z) = \frac{\varepsilon_N}{\kappa z^2}(z'E_2 + y'E_3),$$

and the binormal vector of γ is

$$B = T \wedge N = \frac{\varepsilon_N}{z^3\kappa}\Big((y'^2 - z'^2)E_1 + x'y'E_2 + x'z'E_3\Big).$$

By a direct calculation, we obtain $\nabla_T N$ as follows

$$\nabla_T N = \kappa\Big(\frac{1}{\kappa}\Big)'N - \frac{\varepsilon_N x'y'}{\kappa z^3}E_1 + \frac{\varepsilon_N}{\kappa}\Big[\Big(\frac{z'}{z^2}\Big)' - \frac{y'^2}{z^3}\Big]E_2 + \frac{\varepsilon_N}{\kappa}\Big[\Big(\frac{y'}{z^2}\Big)' - \frac{y'z'}{z^3}\Big]E_3.$$

Using the second Frenet equation (2.2) and the binormal vector B, we have the torsion τ as

$$\tau = g(\nabla_T N, B) = \frac{x'(z'y'' - y'z'')}{\kappa^2 z^5}.$$

From (3.9), we can write $\kappa^2 z^2 = \varepsilon_N(v_1^2 z^2 - \varepsilon_T)$. So, we have

$$\tau = g(\nabla_T N, B) = \frac{\varepsilon_N x'(z'y'' - y'z'')}{z^3(v_1^2 z^2 - \varepsilon_T)}.$$

On the other hand, we have (3.5)-(3.7) because γ is a Killing magnetic curve. Using these equations together with (3.1) and (3.2) we obtain that

$$\tau = g(\nabla_T N, B) = \frac{\varepsilon_N v_1\Big[\Big(\varepsilon_T(2v_2+1) - 2v_1^2\Big)z^2 + \varepsilon_T\Big]}{2(v_1^2 z^2 - \varepsilon_T)}.$$

This completes the proof. □

Note that the space-like Killing magnetic line $\gamma(s) = (\pm\sqrt{2}s, s, 1)$ and the time-like Killing magnetic line $\gamma(s) = (0, -s, 1)$ given in Theorem 3.1 lies in the horosphere $z = 1$. They are horocycles in $\mathcal{H}_1^3$, and from the formulas above, it follows that their curvature $\kappa = 1$ and torsion $\tau = 0$. Also, considering the z-components of the particular nonplanar curves given in Theorem 3.1, we see that the ratio $\kappa(s)/\tau(s)$ is not constant, which means these curves are not helices.

Theorem 3.3 *A smooth light-like curve $\gamma(s)$ in $\mathcal{H}_1^3$ is a Killing magnetic curve corresponding to the Killing vector field $V_1 = \partial_x$ if the component functions $x(s), y(s), z(s)$ of $\gamma(s)$ satisfy the following equations:*

$$x'(s) = v_1 z^2(s), \tag{3.10}$$

$$y'(s) = (v_2 + 1/2)z^2(s) - 1/2, \tag{3.11}$$

$$z'(s) = \pm\sqrt{[(v_2+1/2)^2 - v_1^2]z^4(s) - (v_2+1/2)z^2(s) + 1/4} \tag{3.12}$$

with the initial conditions $x(0) = y(0) = 0, z(0) = 1, x'(0) = v_1, y'(0) = v_2, z'(0) = v_3$ such that $v_1^2 - v_2^2 + v_3^2 = 0$.

In particular, some solutions of these differential equations yield the followings:

1) $\gamma(s) = \Big(0, -\frac{s}{2}, \frac{s}{2} + 1\Big)$, *where* $s > -2$,

2) $\gamma(s) = \Big(0, 1 - \tan(\frac{s}{2} + \frac{\pi}{4}), \tan(\frac{s}{2} + \frac{\pi}{4})\Big)$, *where* $-\frac{\pi}{2} < s < \frac{\pi}{2}$,

3) $\gamma(s) = \Big(\frac{1}{8}(s + \sin s), \frac{1}{8}(-3s + \sin s), \cos\frac{s}{2}\Big)$, *where* $-\pi < s < \pi$,

4) $\gamma(s) = \Big(\frac{\sqrt{2}}{4} - \frac{s}{8} + \lambda(s), \frac{\sqrt{2}}{4} - \frac{3s}{8} - \lambda(s), \sqrt{2}\sinh\frac{s}{2} + \cosh\frac{s}{2}\Big)$,

where $\lambda(s) = \frac{3}{8}\sinh s + \frac{\sqrt{2}}{4}\cosh s$ *and* $s > -2\tanh^{-1}(\frac{1}{\sqrt{2}})$.

Proof Let $\gamma(s) = (x(s), y(s), z(s))$ be a light-like curve in $\mathcal{H}_1^3$. The equations (3.10)-(3.12) are obtained as in the proof of Theorem 3.1 by taking $\varepsilon_T = 0$ with the initial conditions $x(0) = y(0) = 0, z(0) = 1, x'(0) = v_1, y'(0) = v_2$, and $z'(0) = v_3$ such that $v_1^2 - v_2^2 + v_3^2 = 0$.

For $z = 1$, it can be seen easily that there is no light-like Killing magnetic curve in $\mathcal{H}_1^3$ corresponding to the Killing vector field $V_1 = \partial_x$.

Now, we consider some particular solutions of (3.10), (3.11) and (3.12) for some choices of v_1, v_2, and v_3.

Case 1. $v_1 = 0$, $v_2 = -1/2$, and $v_3 = 1/2$. Then, we have $z'(s) = 1/2$ from (3.12) that gives $z(s) = 1 + \frac{s}{2}$ with $s > -2$. From (3.10) and (3.11) we obtain $x(s) = 0$ and $y(s) = -\dfrac{s}{2}$. Therefore, we obtain 1) in Theorem 3.3.

Case 2. $v_1 = 0$, $v_2 = -1$, and $v_3 = 1$. Then, we have $z' = \frac{1}{2}(1 + z^2)$ from (3.12). By integrating this with the initial conditions, we obtain $z(s) = \tan(\frac{s}{2} + \frac{\pi}{4})$, where $-\pi/2 < s < \pi/2$ as $z > 0$. From (3.10) and (3.11), we have $x(s) = 0$ and $y(s) = 1 - \tan(\frac{s}{2} + \frac{\pi}{4})$. Thus, we obtain the light-like curve 2) in Theorem 3.3.

Case 3. $v_1 = \frac{1}{4}$, $v_2 = -\frac{1}{4}$, and $v_3 = 0$. Then, from (3.12) we have $z' = \frac{1}{2}\sqrt{1 - z^2}$ from which and the initial conditions, we find $z(s) = \cos(\frac{s}{2})$, where $\pi < s < \pi$, and using this we obtain from (3.10) and (3.11) that $x(s) = \frac{1}{8}(s + \sin s)$ and $y(s) = \frac{1}{8}(-3s + \sin s)$. Therefore, we obtain the light-like curve 3) in Theorem 3.3.

Case 4. $v_1 = \frac{1}{4}$, $v_2 = -\frac{3}{4}$, and $v_3 = 1/\sqrt{2}$. Then, from (3.12), we have $z' = \frac{1}{2}\sqrt{1 + z^2}$, and its solution is given in (3.8). So, we have from (3.10) that

$$x'(s) = \frac{1}{4}\Big(1 + 3\sinh^2(\frac{s}{2}) + \sqrt{2}\sinh s\Big)$$

that yields $x(s) = \dfrac{\sqrt{2}}{4} - \dfrac{s}{8} + \dfrac{3}{8}\sinh s + \dfrac{\sqrt{2}}{4}\cosh s$. Similarly, we obtain from (3.11) that $y(s) = \dfrac{\sqrt{2}}{4} - \dfrac{3s}{8} - \dfrac{3}{8}\sinh s - \dfrac{\sqrt{2}}{4}\cosh s$. Therefore, we have the light-like curve 4) in Theorem 3.3. □

Now, we will find the torsion τ of a light-like Killing magnetic curve. We know that for a light-like curve $\gamma(s)$ in the above settings, s is an arbitrary parameter for γ. We can reparameterize $\gamma(s)$ in terms of the pseudo-arc length parameter t if $g(\nabla_{\gamma'(s)}\gamma'(s), \nabla_{\gamma'(s)}\gamma'(s)) \neq 0$ along γ.

Let $s = \lambda(t)$, and $\beta(t) = \gamma(\lambda(t))$ be a reparameterization of $\gamma(s)$. Then, we have

$$\dot{\beta}(t) = \frac{d\beta(t)}{dt} = \frac{d\gamma(s)}{ds}\frac{d\lambda(t)}{dt} = \dot{\lambda}(t)\gamma'(s)$$

and

$$\nabla_{\dot{\beta}(t)}\dot{\beta}(t) = \ddot{\lambda}(t)\gamma'(s) + \dot{\lambda}^2(t)\nabla_{\gamma'(s)}\gamma'(s).$$

We choose λ such that $g(\nabla_{\beta'(t)}\beta'(t), \nabla_{\beta'(t)}\beta'(t)) = 1$ which implies $\dfrac{ds}{dt} = \dot{\lambda}(t) = \dfrac{1}{\sqrt{\|\nabla_{\gamma'}\gamma'\|}}$. Then, we have the pseudo-arc length parameter $t = \int_0^s \sqrt{\|\nabla_{\gamma'(u)}\gamma'(u)\|}du$.

Corollary 3.4 *Let $\gamma(s) = (x(s), y(s), z(s))$ be a smooth nongeodesic light-like Killing magnetic curve with arbitrary parameter s corresponding to the Killing vector field $V_1 = \partial_x$ in $\mathcal{H}_1^3$ with the initial conditions $x(0) = y(0) = 0, z(0) = 1, x'(0) = v_1 > 0, y'(0) = v_2$, and $z'(0) = v_3$ such that $v_1^2 - v_2^2 + v_3^2 = 0$. Then, the torsion τ of $\gamma(s)$ is given by*

$$\tau = \frac{2 - (2v_2 + 1)z^2}{2v_1 z^2}. \tag{3.13}$$

Proof Let $\gamma(s)$ be a curve stated in the hypothesis. Then, by using (2.7) with $\varepsilon_T = 0$ and (3.4), we can have

$$\nabla_{\gamma'(s)}\gamma'(s) = \frac{z'}{z^2}E_2 + \frac{y'}{z^2}E_3 \quad \text{and} \quad \|\nabla_{\gamma'(s)}\gamma'(s)\|^2 = \frac{{y'}^2 - {z'}^2}{z^4} = \frac{{x'}^2}{z^4}.$$

As γ is a Killing magnetic curve, using (3.10) we obtain that $\|\nabla_{\gamma'(s)}\gamma'(s)\| = v_1$. Therefore, the pseudo-arc length parameter is $t = \sqrt{v_1}s$, and we can reparameterize the curve $\gamma(s)$ as $\beta(t) = \gamma(t/\sqrt{v_1})$. Hence, we determine the Cartan frame $\{T, N, B\}$ as follows:

$$T = \frac{d\beta(t)}{dt} = \frac{d\gamma(s)}{ds}\frac{ds}{dt} = \frac{1}{\sqrt{v_1}}\Big(\frac{x'}{z}E_1 + \frac{y'}{z}E_2 + \frac{z'}{z}E_3\Big),$$

$$N = \nabla_{\beta'(t)}\beta'(t) = \frac{1}{v_1}\nabla_{\gamma'}(s)\gamma'(s) = \frac{1}{v_1}\Big(\frac{z'}{z^2}E_2 + \frac{y'}{z^2}E_3\Big),$$

$$B = \frac{1}{2v_1^{3/2}z^3}\Big(x'E_1 - y'E_2 - z'E_3\Big).$$

Also, by a direct calculation, we find $\nabla_T N$ as follows

$$\nabla_T N = \frac{1}{v_1^{3/2}}\Big(- \frac{x'y'}{z^3}E_1 + \Big[\Big(\frac{z'}{z^2}\Big)' - \frac{{y'}^2}{z^3}\Big]E_2 + \Big[\Big(\frac{y'}{z^2}\Big)' - \frac{y'z'}{z^3}\Big]E_3\Big). \tag{3.14}$$

Using the second Frenet equation (2.5) and the binormal vector B, we obtain the torsion as $\tau = g(\nabla_T N, B) = \frac{1-2y'}{2v_1 z^2}$. Using (3.11), we get (3.13). □

3.2. The Killing magnetic curves in $\mathcal{H}_1^3$ corresponding to V_2

Theorem 3.5 *A smooth space-like or time-like unit speed curve $\gamma(s)$ in $\mathcal{H}_1^3$ is a normal Killing magnetic curve corresponding to the Killing vector field $V_2 = \partial_y$ if $\gamma(s)$ is one of the followings:*

1) the space-like Killing magnetic line $\gamma(s) = (-s, 0, 1)$ or

2) the time-like Killing magnetic line $\gamma(s) = (s, \pm\sqrt{2}s, 1)$ or

3) the component functions $x(s), y(s), z(s)$ of $\gamma(s)$ satisfy the following equations:

$$x'(s) = (v_1 + 1/2)z^2(s) - 1/2, \tag{3.15}$$

$$y'(s) = v_2 z^2(s), \tag{3.16}$$

$$z'(s) = \pm\sqrt{[v_2^2 - (v_1+1/2)^2]z^4(s) + (\varepsilon_T + v_1 + 1/2)z^2(s) - 1/4} \tag{3.17}$$

with the initial conditions $x(0) = y(0) = 0, z(0) = 1, x'(0) = v_1, y'(0) = v_2$, and $z'(0) = v_3$ such that $v_1^2 - v_2^2 + v_3^2 = \varepsilon_T$.

In particular, some solutions of these differential equations yield the followings:

a) Space-like Killing magnetic curve:

$$\gamma(s) = \Big(- \tfrac{1}{8}(7s + 3\sinh s), -\tfrac{3}{8}(s + \sinh s), \cosh\tfrac{s}{2}\Big), \text{ where } |s| < \infty.$$

b) Time-like Killing magnetic curve:

$$\gamma(s) = \Big(\tfrac{s}{8} + \tfrac{5}{8}\sinh s, \tfrac{5}{8}(s + \sinh s), \cosh\tfrac{s}{2}\Big), \text{ where } |s| < \infty.$$

Proof Let $\gamma(s) = (x(s), y(s), z(s))$ be a unit speed space-like or time-like curve in $\mathcal{H}_1^3$. Then, using (2.11) and (2.12), $\gamma(s)$ is a normal Killing magnetic curve corresponding to the Killing vector field $V_2 = \partial_y$ if $\gamma(s)$ satisfies

$$\nabla_{\gamma'}\gamma' = \partial_y \wedge (x'(s)\partial_x + y'(s)\partial_y + z'(s)\partial_z) = \frac{z'}{z}\partial_x - \frac{x'}{z}\partial_z$$

from which and (2.8), we obtain the following system of differential equations:

$$zx'' = 2x'z' + z', \tag{3.18}$$

$$zy'' = 2y'z', \tag{3.19}$$

$$zz'' = {y'}^2 + {z'}^2 - {x'}^2 - x'. \tag{3.20}$$

We solve this system of differential equations with the initial conditions $x(0) = y(0) = 0, z(0) = 1, x'(0) = v_1, y'(0) = v_2,$ and $z'(0) = v_3$ such that $v_1^2 - v_2^2 + v_3^2 = \varepsilon_T$.

Let $z = 1$. Then, we have $z' = 0$, that is, $v_3 = 0$, and $v_1^2 - v_2^2 = \varepsilon_T$. Also, from (2.7) and (3.20) we get $x' = -\varepsilon$, that is, $x(s) = -\varepsilon_T s$ with $v_1 = -\varepsilon_T$, and from (2.7) we have $y(s) = \pm\sqrt{1-\varepsilon_T}s$ with $v_2 = \pm\sqrt{1-\varepsilon_T}$. Therefore, we obtain 1) when $\varepsilon_T = 1$, and 2) when $\varepsilon_T = -1$ in Theorem 3.5.

Now, we suppose that $z(s)$ is a nonconstant function on some open interval containing 0. From the first integrals of (3.18) and (3.19), we obtain (3.15) and (3.16), respectively. By taking the derivative (2.7) with respect to s and using (2.7), (3.18) and (3.19), we can have (3.20), that is, a solution of (2.7), (3.18) and (3.19) is a solution of (3.20). Therefore, from (2.7), (3.15) and (3.16) we obtain (3.17).

Now, we consider a particular solution of (3.15), (3.16) and (3.17) for $v_1 = -\varepsilon_T - 1/4$, $v_2 = 1/4 - \varepsilon_T$ and $v_3 = 0$. Then, from (3.17) we have $z' = \frac{1}{2}\sqrt{z^2-1}$ which yields $z(s) = \cosh(\frac{s}{2})$, where $|s| < \infty$, and from (3.15) and (3.16) we have

$$x(s) = -\frac{4\varepsilon_T + 3}{8}s + \frac{1-4\varepsilon_T}{8}\sinh s \quad \text{and} \quad y(s) = \frac{1-4\varepsilon_T}{8}(s + \sinh s),$$

respectively. Thus, for $\varepsilon_T = 1$ and $\varepsilon_T = -1$ we obtain a) and b) in Theorem 3.5, respectively. □

Corollary 3.6 *Let $\gamma(s) = (x(s), y(s), z(s))$ be a smooth space-like or time-like unit speed Killing magnetic curve corresponding to the Killing vector field $V_1 = \partial_y$ in $\mathcal{H}_1^3$ with the initial conditions $x(0) = y(0) = 0, z(0) = 1, x'(0) = v_1, y'(0) = v_2,$ and $z'(0) = v_3$ such that $v_1^2 - v_2^2 + v_3^2 = \varepsilon_T$. Then, the curvature κ and the torsion τ of $\gamma(s)$ are given by*

$$\kappa = \sqrt{\varepsilon_N\left(v_2^2 + \frac{\varepsilon_T}{z^2}\right)} \quad \textit{and} \quad \tau = \frac{\varepsilon_N v_2\Big((\varepsilon_T(2v_1+1) + 2v_2^2)z^2 + \varepsilon_T\Big)}{2(v_2^2 z^2 + \varepsilon_T)}.$$

Since the proof of this corollary is similar to the proof of Corollary 3.2 we omit the proof here.

The space-like Killing magnetic line $\gamma(s) = (-s, 0, 1)$ and the time-like Killing magnetic line $\gamma(s) = (s, \pm\sqrt{2}s, 1)$ given in Theorem 3.5 are horocycles, and they lie in the horosphere $z = 1$. The other two particular curves given in Theorem 3.5 are not helices because the ratio κ/τ for these curves is not constant.

For light-like Killing magnetic curves $\gamma(s)$ in $\mathcal{H}_1^3$ corresponding to the Killing vector field $V_2 = \partial_y$, we can deduce the following result by taking $\varepsilon_T = 0$ in Theorem 3.5 and in its proof.

Theorem 3.7 *A smooth light-like curve $\gamma(s)$ in $\mathcal{H}_1^3$ is the Killing magnetic curve corresponding to the Killing vector field $V_2 = \partial_y$ if the component functions $x(s), y(s), z(s)$ of $\gamma(s)$ satisfy the following equations:*

$$x'(s) = (v_1 + 1/2)z^2(s) - 1/2, \tag{3.21}$$

$$y'(s) = v_2 z^2(s), \tag{3.22}$$

$$z'(s) = \pm\sqrt{[v_2^2 - (v_1 + 1/2)^2]z^4(s) + (v_1 + 1/2)z^2(s) - 1/4} \tag{3.23}$$

with the initial conditions $x(0) = y(0) = 0, z(0) = 1, x'(0) = v_1, y'(0) = v_2, z'(0) = v_3$ such that $v_1^2 - v_2^2 + v_3^2 = 0$.

A particular solution of these differential equations produces the light-like Killing magnetic curve:

$\gamma(s) = \left(\frac{1}{8}(-3s + \sinh s), \frac{1}{8}(s + \sinh s), \cosh\frac{s}{2}\right)$, *where $|s| < \infty$.*

From this theorem, we state the following corollary similar to Corollary 3.4 without proof.

Corollary 3.8 *Let $\gamma(s) = (x(s), y(s), z(s))$ be a smooth nongeodesic light-like Killing magnetic curve with arbitrary parameter s corresponding to the Killing vector field $V_1 = \partial_y$ in $\mathcal{H}_1^3$ with the initial conditions $x(0) = y(0) = 0, z(0) = 1, x'(0) = v_1 > 0, y'(0) = v_2$, and $z'(0) = v_3$ such that $v_1^2 - v_2^2 + v_3^2 = 0$. Then, the torsion τ of $\gamma(s)$ is given by*

$$\tau = \frac{(2v_1 + 1)z^2 - 2}{2v_2 z^2}.$$

3.3. The Killing magnetic curves in $\mathcal{H}_1^3$ corresponding to V_3

Theorem 3.9 *A smooth space-like or time-like unit speed curve $\gamma(s)$ in $\mathcal{H}_1^3$ is a normal Killing magnetic curve corresponding to the Killing vector field $V_3 = x\partial_x + y\partial_y + z\partial_z$ if the component functions $x(s), y(s), z(s)$ of $\gamma(s)$ satisfy the following system of differential equations:*

$$zx'' = 2x'z' + yz' - zy', \tag{3.24}$$

$$zy'' = 2y'z' + xz' - zx', \tag{3.25}$$

$$zz'' = 2z'^2 + xy' - yx' - \varepsilon_T z^2. \tag{3.26}$$

Two particular solutions of these differential equations yield the following Killing magnetic curves:

1) $\gamma(s) = (v_1 \sinh s - v_2 \cosh s,\ v_2 \sinh s - v_1 \cosh s,\ 1)$, which is the reparameterization of the hyperbola $y^2 - x^2 = \varepsilon_T$ in the plane $z = 1$, where $v_1^2 - v_2^2 = \varepsilon_T$ and $|s| < \infty$. The curve $\gamma(s)$ is space-like if $\varepsilon_T = 1$, and time-like if $\varepsilon_T = -1$.

2) The space-like curve $\gamma(s) = \left(\frac{2v_0 e^{s/2}}{\sqrt{5}}\sinh\left(\frac{\sqrt{5}s}{2}\right), \frac{2v_0 e^{s/2}}{\sqrt{5}}\sinh\left(\frac{\sqrt{5}s}{2}\right), e^s\right)$, where $|s| < \infty$, and $v_0 \in \mathbb{R}$. If $v_0 = 0$, the space-like curve $\gamma(s) = \left(0, 0, e^s\right)$ is a geodesic of $\mathcal{H}_1^3$.

Proof Let $\gamma(s) = (x(s), y(s), z(s))$ be a unit speed space-like or time-like curve in $\mathcal{H}_1^3$. Then, using (2.11) and (2.12), $\gamma(s)$ is a normal Killing magnetic curve corresponding to the Killing vector field $V_3 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}$

if $\gamma(s)$ satisfies

$$\begin{aligned}\nabla_{\gamma'}\gamma' &= (x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}) \wedge (x'(s)\partial_x + y'(s)\partial_y + z'(s)\partial_z) \\ &= \frac{yz' - zy'}{z}\partial_x + \frac{xz' - zx'}{z}\partial_y + \frac{xy' - yx'}{z}\partial_z\end{aligned}$$

from which and (2.8), we obtain (3.24)-(3.26) by using (2.7).

We consider two particular solutions of equations (3.24)-(3.26).

Case 1. $z = 1$. Then, we get from (3.24)-(3.26) the system of differential equations as $x'' = -y'$, $y'' = -x'$, $xy' - yx' = \varepsilon_T$. The solution of this system under the initial conditions $x(0) = -v_2, y(0) = -v_1, x'(0) = v_1, y'(0) = v_2$ with $v_1^2 - v_2^2 = \varepsilon_T$ produces $x(s) = v_1 \sinh s - v_2 \cosh s, \quad y(s) = v_2 \sinh s - v_1 \cosh s$. Therefore, we have the curve 1) in Theorem 3.9.

Case 2. We solve differential equations (3.24)-(3.26) for $y(s) = x(s)$ and $\varepsilon_T = 1$ together with the initial conditions $x(0) = y(0) = 0, z(0) = 1, x'(0) = y'(0) = v_0$, and $z'(0) = 1$. As $y(s) = x(s)$, from (2.7) we have $z' = z$ that gives $z = e^s$. This also holds (3.26). Considering $z = e^s$ and $y(s) = x(s)$, the equations (3.24) and (3.25) yield the differential equation of the form $x'' - x' - x = 0$ whose solution is

$$x(s) = y(s) = \frac{2v_0}{\sqrt{5}} e^{s/2} \sinh\Big(\frac{\sqrt{5}s}{2}\Big).$$

Therefore, we have the curve 2) in Theorem 3.9. When we take $v_0 = 0$, we have $\gamma(s) = (0, 0, e^s)$ in 3) which is a vertical geodesic of $\mathcal{H}_1^3$. □

Theorem 3.10 *A smooth light-like curve $\gamma(s)$ in $\mathcal{H}_1^3$ is a Killing magnetic curve corresponding to the Killing vector field $V_3 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}$ if the component functions $x(s), y(s), z(s)$ of $\gamma(s)$ satisfy the following system of differential equations:*

$$zx'' = 2x'z' + yz' - zy', \tag{3.27}$$

$$zy'' = 2y'z' + xz' - zx', \tag{3.28}$$

$$zz'' = 2z'^2 + xy' - yx'. \tag{3.29}$$

A particular solution of these differential equations yields $\gamma(s) = (v_0(1 - e^{-s}), \mp v_0(1 - e^{-s}), 1)$ which is the reparameterization of light-like lines in the plane $z = 1$, where v_0 is a nonzero constant.

Proof Let $\gamma(s) = (x(s), y(s), z(s))$ be a light-like curve in $\mathcal{H}_1^3$, that is, $\varepsilon_T = 0$. The proof is similar to the proof of Theorem 3.9, and equations (3.27)-(3.29) follows from (3.24)-(3.26) by taking $\varepsilon_T = 0$ in (3.26).

Let $z = 1$. Then, from (3.27)-(3.29) we have $x'' = -y'$, $y'' = -x'$ and $xy' - yx' = 0$. And also, from (2.7) we get $y' = \pm x'$. The solution of these differential equations under the initial conditions $x(0) = y(0) = 0, x'(0) = v_0 \neq 0$, $y'(0) = \pm v_0$ gives $x(s) = v_0(1 - e^{-s}), \quad y(s) = \pm v_0(1 - e^{-s})$. Therefore, we have the curve in Theorem 3.10. □

3.4. The Killing magnetic curves in $\mathcal{H}_1^3$ corresponding to V_4

Theorem 3.11 *A smooth space-like or time-like unit speed curve $\gamma(s)$ in $\mathcal{H}_1^3$ is a normal Killing magnetic curve corresponding to the Killing vector field $V_4 = y\partial_x + x\partial_y$ if the component functions $x(s), y(s), z(s)$ of $\gamma(s)$ satisfy the following system of differential equations:*

$$zx'' = (2x' + x)z', \tag{3.30}$$

$$zy'' = (2y' + y)z', \tag{3.31}$$

$$zz'' = 2{z'}^2 + yy' - xx' - \varepsilon_T z^2. \tag{3.32}$$

Two particular solutions of these differential equations yield the following Killing magnetic curves:

1) The space-like curve

$\gamma(s) = \Big(e^s(a\cosh(\sqrt{2}s) + \frac{v-a}{\sqrt{2}}\sinh(\sqrt{2}s)),\ e^s(a\cosh(\sqrt{2}s) + \frac{v-a}{\sqrt{2}}\sinh(\sqrt{2}s)),\ e^s\Big)$, *where* $a, v \in \mathbb{R}$, *and* $|s| < \infty$.

2) The time-like curve $\gamma(s) = \Big(-\dfrac{\sqrt{2}}{\sqrt{c^2-1}}e^{-s}, -\dfrac{c\sqrt{2}}{\sqrt{c^2-1}}e^{-s}, e^{-s}\Big)$, *where* $|s| < \infty$.

Proof Let $\gamma(s) = (x(s), y(s), z(s))$ be a unit speed space-like or time-like curve in $\mathcal{H}_1^3$. Then, using (2.11) and (2.12), $\gamma(s)$ is a normal Killing magnetic curve corresponding to the Killing vector field $V_4 = y\partial_x + x\partial_y$ if $\gamma(s)$ satisfies

$$\begin{aligned}\nabla_{\gamma'}\gamma' &= (y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}) \wedge (x'(s)\partial_x + y'(s)\partial_y + z'(s)\partial_z)\\ &= \frac{xz'}{z}\partial_x + \frac{yz'}{z}\partial_y + \frac{yy' - xx'}{z}\partial_z\end{aligned}$$

from which and (2.8), we obtain (3.30)-(3.32) by using (2.7). Also, from (3.30) and (3.31), we can write the following proportions:

$$\frac{x''}{2x' + x} = \frac{y''}{2y' + y} = \frac{z'}{z} = \lambda. \tag{3.33}$$

This implies that $x(s)$ and $y(s)$ are proportional, that is, $y(s) = cx(s)$ for some nonzero constant c. Hence, using $z' = \lambda z$, we have from (2.7) that

$$(1 - c^2){x'}^2 = (\varepsilon_T - \lambda^2)z^2. \tag{3.34}$$

Note that it can be shown that a solution of (3.30), (3.31) and (2.7) satisfies (3.32).

Now, we obtain two particular solutions of equations (3.30)-(3.32) under some initial conditions.

Case 1. $\varepsilon_T = 1$, $c = 1$ and $\lambda = 1$. Let $x(0) = y(0) = a, z(0) = 1, x'(0) = y'(0) = v$, and $z'(0) = 1$, where $a, v \in \mathbb{R}$. Then, the equation (3.34) is satisfied, and also we obtain from (3.33) that $z(s) = e^s$. Form (3.33), we have the differential equation $x'' = 2x' + x$ whose solution is

$$x(s) = e^s(a\cosh(\sqrt{2}s) + \frac{v-a}{\sqrt{2}}\sinh(\sqrt{2}s))$$

for some integration constants $a, v \in \mathbb{R}$. Considering $y = x$ as $c = 1$, we obtain the space-like curve 1) in Theorem 3.11.

Case 2. $\varepsilon_T = -1$, $c^2 > 1$ and $\lambda = -1$. Let $x(0) = -\sqrt{\frac{2}{c^2-1}}$, $y(0) = -c\sqrt{\frac{2}{c^2-1}}$, $z(0) = 1, x'(0) = \sqrt{\frac{2}{c^2-1}}$, $y'(0) = c\sqrt{\frac{2}{c^2-1}}$, and $z'(0) = -1$. Then, form (3.33) we have $z(s) = e^{-s}$, and also from the equation (3.34) we get $x' = \sqrt{\frac{2}{c^2-1}}e^{-s}$, and using the initial conditions we have $x(s) = -\sqrt{\frac{2}{c^2-1}}e^{-s}$. Considering $y = cx$, it can be shown that $x(s), y(s)$ and $z(s)$ satisfy (3.30)-(3.32). Therefore, we obtain the time-like curve 2) in Theorem 3.11. □

Theorem 3.12 *A smooth light-like curve $\gamma(s)$ in $\mathcal{H}_1^3$ is the Killing magnetic curve corresponding to the Killing vector field $V_4 = y\partial_x + x\partial_y$ if the component functions $x(s), y(s), z(s)$ of $\gamma(s)$ satisfy the following system of differential equations:*

$$zx'' = (2x' + x)z', \tag{3.35}$$

$$zy'' = (2y' + y)z', \tag{3.36}$$

$$zz'' = 2z'^2 + yy' - xx'. \tag{3.37}$$

Two particular solutions of these differential equations yield the following light-like Killing magnetic curves:

1) $\gamma(s) = \Big(as, \pm as, 1\Big)$, *where* $a \in \mathbb{R}\backslash\{0\}$.

2) $\gamma(s) = \Big(\dfrac{e^{-s}}{\sqrt{c^2-1}}, \dfrac{ce^{-s}}{\sqrt{c^2-1}}, e^{-s}\Big)$, *where* $|c| > 1$.

Proof Let $\gamma(s) = (x(s), y(s), z(s))$ be a light-like Killing magnetic curve in $\mathcal{H}_1^3$ corresponding to the Killing vector field V_4. The proof is similar to the proof of Theorem 3.11, and equations (3.35)-(3.37) follows from (3.30)-(3.32) by taking $\varepsilon_T = 0$ in (3.32).

We obtain two particular solutions of equations (3.35)-(3.37). Suppose that $z = 1$. Then, from (3.35)-(3.37) we get $x'' = 0$, $y'' = 0$ and $yy' - xx' = 0$, respectively. Their solutions are $x(s) = as$ and $y(s) = \pm as$ for a nonzero constant a. Thus, we obtain the curve 1) in Theorem 3.12.

Now, we give another solution of (3.35)-(3.37). From (3.35) and (3.36) we have again (3.33) from which we have $z' = \lambda z$, and $x(s)$ and $y(s)$ are proportional, that is, $y(s) = cx(s)$ for some nonzero constant c. Let $c^2 > 1$ and $\lambda = -1$. Considering the initial conditions $x(0) = \frac{1}{\sqrt{c^2-1}}$, $y(0) = \frac{c}{\sqrt{c^2-1}}$, $z(0) = 1, x'(0) = -\frac{1}{\sqrt{c^2-1}}$, $y'(0) = -\frac{c}{\sqrt{c^2-1}}$, and $z'(0) = -1$, we then have $z(s) = e^{-s}$. Also, we have from (2.7) for $\varepsilon_T = 0$ that $x'(s) = \frac{1}{\sqrt{c^2-1}}z'$. Using the initial conditions, we obtain $x(s) = \frac{z}{\sqrt{c^2-1}} = \frac{e^{-s}}{\sqrt{c^2-1}}$. Considering $y = cx$, it can be seen that $x(s), y(s)$ and $z(s)$ satisfy (3.35)-(3.37). Therefore, we obtain the curve 2) in Theorem 3.12. □

3.5. The Killing magnetic curves in $\mathcal{H}_1^3$ corresponding to V_5 and V_6

Theorem 3.13 *A smooth space-like or time-like unit speed curve $\gamma(s)$ in $\mathcal{H}_1^3$ is a normal Killing magnetic curve corresponding to the Killing vector field $V_5 = \frac{1}{2}(x^2 + y^2 - z^2)\partial_x + xy\partial_y + xy\partial_z$ if the component functions*

$x(s), y(s), z(s)$ of $\gamma(s)$ satisfy the following system of differential equations:

$$zx'' = (2x' + xy)z' - xzy', \tag{3.38}$$

$$zy'' = 2y'z' + \frac{1}{2}(x^2 + y^2 - z^2)z' - zxx', \tag{3.39}$$

$$zz'' - 2z'^2 = \frac{1}{2}(x^2 + y^2 - z^2)y' - xyx' - \varepsilon_T z^2. \tag{3.40}$$

Proof Let $\gamma(s) = (x(s), y(s), z(s))$ be a unit speed space-like or time-like curve in $\mathcal{H}_1^3$. Then, using (2.11) and (2.12), $\gamma(s)$ is a normal Killing magnetic curve corresponding to the Killing vector field $V_5 = \frac{1}{2}(x^2 + y^2 - z^2)\partial_x + xy\partial_y + xy\partial_z$ if $\gamma(s)$ satisfies

$$\begin{aligned}\nabla_{\gamma'}\gamma' &= V_5 \wedge (x'(s)\partial_x + y'(s)\partial_y + z'(s)\partial_z) \\ &= \frac{x(yz' - zy')}{z}\partial_x + \frac{(x^2 + y^2 - z^2)z' - 2zxx'}{2z}\partial_y + \frac{(x^2 + y^2 - z^2)y' - 2yxx'}{2z}\partial_z\end{aligned}$$

from which we obtain (3.38)-(3.40) by using (2.7) and (2.8). □

Theorem 3.14 *A smooth light-like curve $\gamma(s)$ in $\mathcal{H}_1^3$ is a Killing magnetic curve corresponding to the Killing vector field $V_5 = \frac{1}{2}(x^2 + y^2 - z^2)\partial_x + xy\partial_y + xy\partial_z$ if the component functions $x(s), y(s), z(s)$ of $\gamma(s)$ satisfy the following system of differential equations:*

$$zx'' = (2x' + xy)z' - xzy',$$

$$zy'' = 2y'z' + \frac{1}{2}(x^2 + y^2 - z^2)z' - zxx',$$

$$zz'' - 2z'^2 = \frac{1}{2}(x^2 + y^2 - z^2)y' - xyx'.$$

The proof follows from Theorem 3.13 by taking $\varepsilon_T = 0$ in (3.40).

By a similarly way, we can have the followings for the Killing vector field V_6.

Theorem 3.15 *A smooth space-like or time-like unit speed curve $\gamma(s)$ in $\mathcal{H}_1^3$ is a normal Killing magnetic curve corresponding to the Killing vector field $V_6 = xy\partial_z + \frac{1}{2}(x^2 + y^2 + z^2)\partial_y + yz\partial_z$ if the component functions $x(s), y(s), z(s)$ of $\gamma(s)$ satisfy the following system of differential equations:*

$$zx'' = 2x'z' + \frac{1}{2}(x^2 + y^2 + z^2)z' - zyy',$$

$$zy'' = 2y'z' + xyz' - yzx',$$

$$zz'' - 2z'^2 = xyy' - \frac{1}{2}(x^2 + y^2 + z^2)x' - \varepsilon_T z^2.$$

Theorem 3.16 *A smooth light-like curve $\gamma(s)$ in $\mathcal{H}_1^3$ is the Killing magnetic curve corresponding to the Killing vector field $V_6 = xy\partial_z + \frac{1}{2}(x^2 + y^2 + z^2)\partial_y + yz\partial_z$ if the component functions $x(s), y(s), z(s)$ of $\gamma(s)$ satisfy the* *following system of differential equations:*

$$x'' = \frac{2x'z' + \frac{1}{2}(x^2+y^2+z^2)z' - zyy'}{z},$$

$$y'' = \frac{2y'z' + xyz' - yzx'}{z},$$

$$zz'' - 2z'^2 = xyy' - \frac{1}{2}(x^2+y^2+z^2)x'.$$

Conflict of interest

The author declares no potential conflict of interest.

Data availability statement

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

Funding

Not applicable.

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