HEAD-ON COLLISIONS OF SOLITARY WAVES

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Abstract

The interaction of solitary waves in various physical media is a long time studied subject in nonlinear wave theory. For overtaking collision between solitary waves, one can use the inverse scattering transform method to obtain the overtaking colliding effect of solitary waves. However, for the head-on collision between solitary waves, one must employ some kind of asymptotic expansion to solve the original field equations.

This thesis addresses head-on collision problem between two solitary waves. The head-on collision of solitary waves in shallow water is re-examined upon discovering the wrongness of the statement about the secular terms in the pioneering work of Su and Mirie (J. Fluid Mech., 98:509-525, 1980). In the first part, based on the above argument, the head-on collision of two solitary waves propagating in shallow water is studied by introducing a set of stretched coordinates that includes some unknown trajectory functions which are to be determined so as to remove secularities that might occur in the solution. Expanding the field variables and trajectory functions into power series, a set of differential equations governing various terms in the perturbation expansion is obtained. By solving them under non-secularity condition, the evolution equations and also the expressions for phase shifts are determined. As opposed to the result of previous studies our calculation shows that the phase shifts depend on amplitudes of both colliding waves. In the second part, the head-on-collision of solitary waves in shallow water theory is examined through the use of extended Poincaré-Lighthill-Kuo(PLK) method. Following a similar procedure with the previous part, the speed correction terms and the trajectory functions are determined. The result obtained here is exactly same with that found in the first part. In the third part, the head-on collision of the solitary waves in fluid-filled elastic tubes is studied by employing the extended PLK method. Pursuing the procedure in the previous part, the speed correction terms and the trajectory functions are obtained. The results of our calculation show that both the evolution equations and the phase shifts are quite different from those of Xue (Phys. Lett. A, 331:409-413, 2004). As opposed to the result of previous works on the same subject, the phase shifts depend on the amplitudes of both colliding waves.

SOLİTER DALGALARIN KAFA KAFAYA ÇARPIŞMASI

Özet

Nonlineer dalga teorisinde soliter dalgaların çeşitli fiziksel ortamlardaki etkileşimi uzun zamandır üzerinde çalışılan bir konudur. Soliter dalgaların aynı yönde hareket ederken birinin diğerini yakalayarak çarpışması durumunda, bu çarpışmanın etkilerini ortaya çıkarmak için ters saçılım dönüşümü(inverse scattering transform) yöntemi kullanılabilir. Ancak, zıt yönde hareket eden soliter dalgaların kafa kafaya çarpışması durumunda alan denklemlerini çözmek için bir çeşit asimptotik açılım kullanılmalıdır.

Bu tezde iki soliter dalganin kafa kafaya çarpışması problemi incelenecektir. Su ve Mirie'nin (J. Fluid Mech., 98:509-525, 1980) öncülük eden çalışmasındaki seküler terimlerle ilgili ifadelerinin yanlışlığının tarafımızdan ortaya çıkarılması sonucunda, sığ sudaki yalnız dalgaların kafa kafaya çarpışması problemi yeniden incelenmiştir. İlk kısımda, yukarıdaki argümanı temel alarak, sığ suda yayılan iki soliter dalganın kafa kafaya çarpışması problemi ele alınmıştır. Bunun için çözüm sırasında meydana gelebilecek seküler terimlerin ortadan kaldırılmasıyla belirlenebilecek bilinmeyen bazı yörünge fonksiyonları içeren gerilmiş koordinatlar kullanılmıştır. Alan değişkenlerini ve yörünge fonksiyonlarını kuvvet serilerine açarak, pertürbasyon açılımındaki çeşitli terimleri yöneten diferansiyel denklemler elde edilmiştir. Çözümlerin seküler terim içermeme koşulu altında evolüsyon denklemleri ve faz farklarının ifadeleri bulunmuştur. Hesaplamalar sonucunda, daha önceki çalışmaların aksine faz farklarının çarpışan her iki dalganın genliğine bağlı olduğu görülmüştür. İkinci kısımda, sığ su teorisinde soliter dalgaların kafa kafaya çarpışması problemi, genişletilmiş Poincaré-Lighthill-Kuo(PLK) yöntemi kullanılarak incelenmiştir. Bir önceki kısımla benzer bir yol takip edilerek hız düzeltme terimleri ve yörünge fonksiyonları elde edilmiştir. Burada bulunan sonuçların ilk kısımda bulunan sonuçlarla tamamen aynı olduğu görülmüştür. Üçüncü kısımda ise genişletilmiş PLK yöntemi kullanılarak akışkan ile dolu elastik tüplerde soliter dalgaların kafa kafaya çarpışması problemi incelenmiştir. Önceki kısımda uygulanan yöntem takip edilerek hız düzeltme terimleri ve yörünge fonksiyonları elde edilmiştir. Sonuç olarak, elde edilen evolüsyon denklemlerinin ve faz farklarının Xue'nin (Phys. Lett. A, 331:409-413, 2004) çalışmasındakilerden farklı olduğu tespit edilmiştir. Aynı konuda daha önce yapılan çalışmaların aksine faz farklarının çarpışan dalgaların her ikisinin genliğine bağlı olduğu gösterilmiştir.

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Chapter 1

Introduction

The interaction of solitary waves in various physical media is a long time studied subject in nonlinear wave theory. The interaction problem have attracted considerable amount of interest and curiosity whether the process is elastic or not since the introduction of the concept of solitary wave. The study of solitary waves began with the observations by J. Scott Russell [1, 2] over a century ago. Russell introduced the concept of solitary wave with the following description [2]: "*a large solitary elevation, a rounded, smooth and well defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed... Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon*". Russell did extensive experiments in a laboratory scale wave tank in order to study this phenomenon and further investigations were undertaken by several researchers to understand this phenomenon. In 1895 Korteweg and deVries [3] provided a simple analytic foundation for the study of solitary waves by developing an equation, now known as KdV equation, for shallow water waves. Their equation had permanent wave solutions, including solitary waves.

One of the important behaviors of solitary waves is the interaction of multiple solitary waves. Interactions of solitary waves can be classified as head-on collisions (the counterpropagating waves) and overtaking collisions (the copropagating waves). The overtaking collision is also called as strong interaction which is the consequence of a relatively long interaction time and the head-on collision of solitary waves is called weak interaction, instead, owing to its relatively short interaction time. In 1965, Zabusky and Kruskal [4] discovered numerically that the interaction of two solitary waves is elastic in their study of the continuum approximation to the nonlinear discrete mass string of Fermi-Pasta-Ulam [5]. Until this time, it was generally supposed that if two solitary waves collided, the nonlinear interaction upon collision would completely destroy their integrity and identity. Two solitary waves with different amplitudes propagating in the same direction can collide, exchange their energies and position with one another, and, then separate off, regaining their original forms. Throughout the whole process of the collision, the solitary waves are remarkably stable entities preserving their identities through the interaction. Each solitary wave reemerges from the collision retaining its original identity, except for a phase shift, that is, they are shifted in position relative to where they would have been had no interaction occurred. The taller (faster) wave is shifted to the right and the shorter (slower) to the left. The features of the overtaking collision, that were mentioned here, had been revealed by the foundation of the inverse scattering transform (IST) method which provides solution to the KdV equation. Since IST gives solutions which are KdV solitary waves that travel in the same direction under the boundary conditions vanishing at infinity, it can be used to discover the effects of overtaking collision between solitary waves.

However, for the head-on collision case, some kind of numerical or asymptotic method should be employed to study the collision of the solitary waves propagating in the opposite direction. As is shown in the next chapter, there are various studies in this regard including Su and Mirie's fundamental study [6] in which they employed an asymptotic expansion called the Poincaré-Lighthill-Kuo (PLK) method. Based on Su and Mirie's approach, the head-on collision problems in various physical media had been studied by several researchers from the time the article [6] was published to present with an increasing amount of interest. This situation motivated us to study the head-on collision problem and in this manner we examined the Su and Mirie's approach to the problem of the head-on collision of two solitary waves propagating in the shallow water. Su and Mirie [6] introduced a set of stretched coordinates where some unknown trajectory functions were presented. These unknown functions, characterizing the phase shifts after collision, are to be determined from the requirements of non-secular solutions of the field quantities. By expanding the field variables and the unknown trajectory functions, they obtained a set of differential equations. They tried to obtain solutions to these evolution equations under the restrictions of non-secularity of the solution. They made a statement that "*although this term does not cause any secularity at this order but it will cause the secularity at higher order expansion, therefore, that term must vanish*". As a consequence of this statement, they were able to evaluate the unknown trajectory functions, that is, the phase shifts after the collision had been computed. So this statement is the most interesting part of their analysis. As a result of our calculations for higher order expansion, we observe that this is not the case. This observation constitutes the point of origin of this thesis.

This thesis is organized as follows. In Chapter 2, a review of previous experimental, theoretical and numerical investigations, the derivation of the field equations for shallow water waves and fluid-filled elastic tubes and a review of the reductive perturbation and Poincaré-Lighthill-Kuo methods are summarized. The head-on collision problem between two solitary waves in shallow water is re-examined in view of the above mentioned observation in Chapter 3, which includes evaluation of the surface elevation parameter, the axial velocity parameter and the explicit expressions of phase shifts. Also the differences between the phase shifts obtained in previous studies and the present work are discussed. In Chapter 4, head-on collision between two solitary waves in shallow water is investigated by employing the extended PLK method and the expressions of phase shifts, surface elevation and axial velocity parameters are obtained. The variations of the wave profiles for right-going wave before and after the collision are illustrated. Also, the obtained results are discussed in comparison with the previous chapter and Su and Mirie [6]. In Chapter 5, head-on collision problem of the solitary waves in fluid-filled elastic tubes is studied by using the extended PLK method and the differences between the results of Xue [7], in which the same problem had been studied by using Su and Mirie's approach, and the present work are discussed. Finally overall conclusions are presented in Chapter 6.

Chapter 2

State of the Art and the Summary of Field Equations

2.1 Solitary Waves

The solitary wave, so-called, occurs as this single entity and is localised, and was first observed by J. Scott Russell on the Edinburgh Glasgow canal in 1834; he called it the 'great wave of translation'. Russell reported his observations to the British Association in his 1844 'Report on Waves' [2]. Russell also performed some laboratory experiments, generating solitary waves by dropping a weight at one end of a water channel. He was able to deduce empirically that the speed, c, of the solitary wave is obtained from

$$c^2 = g(h + a), \tag{2.1}$$

where a is the amplitude of the wave, h the undisturbed depth of water and g the acceleration of the gravity. Further investigations were undertaken by Airy [8], Stokes [9],Boussinesq [10] and Rayleigh [11] in an attempt to understand this phenomenon. Both Boussinesq and Rayleigh assumed that a solitary wave has a length scale much greater than the depth of the water. They deduced, from the equations of motion for an inviscid incompressible fluid, Russell's formula for wave speed. In fact they also showed that the wave profile $z = \zeta(x, t)$ is given by

$$\zeta(x, t) = a \ \mathrm{sech}^2\{\beta(x - ct)\}, \tag{2.2}$$

where $\beta^{-2} = 4h^2(h+a)/3a$ for any $a > 0$, although the sech^2 profile is only correct if $a < h$. These investigations provoked much lively discussion and controversy as to whether the inviscid equations of water waves would possess such solitary wave solutions. The issue was finally resolved by Korteweg and de Vries [3]. They derived a nonlinear evolution equation governing long, that is, equilibrium level(depth) is small relative to the water wavelength, one dimensional, small amplitude, surface gravity waves propagating in a shallow water channel

$$\frac{\partial \eta}{\partial \tau} = \frac{3}{2}\sqrt{\frac{g}{h}}\frac{\partial}{\partial \xi}\left(\frac{1}{2}\eta^2 + \frac{2}{3}\alpha\eta + \frac{1}{3}\sigma\frac{\partial^2 \eta}{\partial \xi^2}\right), \quad \sigma = \frac{1}{3}h^3 - Th/(\rho g), \tag{2.3}$$

where η is the surface elevation of the wave above the equilibrium level h, α a small arbitrary constant related to the uniform motion of liquid, g the gravitational constant, T the surface tension and ρ the density (the terms "long" and "small" are meant in comparison to the depth of the channel). The controversy was now resolved since the equation (2.3), now known as the Korteweg-de Vries (KdV) equation, has permanent wave solutions, including solitary wave solutions. Equation (2.3) may be brought into nondimensional form as

$$u_t + 6uu_x + u_{xxx} = 0, \tag{2.4}$$

where subscripts denote partial differentiations. To get the above specific form, we have rescaled and translated the dependent and independent variables in various applications to eliminate the physical constants. Any desired coefficients can be inserted into the equation by such transformations. From the original form of the KdV equation (2.3), the transformations

$$t = \frac{1}{2}\sqrt{\frac{g}{h\sigma}}\tau, \qquad x = -\sigma^{-1/2}\xi, \qquad u = \frac{1}{2}\eta + \frac{1}{3}\alpha$$

give us (2.4). Note that (2.4) is invariant to arbitrary translations in x and t since they appear only in the differentiations. Also, because all derivatives are of odd order, reversing the signs of both x and t does not alter the equation. Moreover, the KdV equation is Galilean invariant, that is, it remains unchanged by the transformation

$$t^{'} = t, \quad x^{'} = x - ct, \quad u^{'}(x^{'}, t^{'}) = u(x,t) - \frac{1}{6}c,$$

where c is some constant. This corresponds to a steady moving reference frame with velocity c.

In spite of this early derivation of the KdV equation, not much progress is made until a new application of the model equation found in the study of collision-free hydromagnetic waves by Gardner and Morikawa [12]. Subsequently the KdV equation has arisen in a number of other physical contexts, stratified internal waves, ion-acoustic waves, plasma physics, lattice dynamics, etc. Kruskal [13] and Zabusky [14–16] showed the KdV equation governs longitudinal waves propagating in a one-dimensional lattice of equal masses coupled by nonlinear springs, the Fermi-Pasta-Ulam problem [5]. Other applications to plasma physics were given by Berezin and Karpman [17] and by Washimi and Taniuti [18] in their study of ion-acoustic waves in a cold plasma. Wijngaarden [19] found it described pressure waves in a liquid-gas bubble mixture. Shen [20] derived the KdV equation in the study of 3-dimensional water waves. Su and Gardner [21] and Taniuti and Wei [22] showed it arises from several general classes of equations. For details and further references see, [23–26].

It has been known for a long time that the KdV equation (2.4) possesses the solitary wave solution of the form

$$u(x,t) = 2a^2\text{sech}^2[a(x - 4a^2t - x_0)], \tag{2.5}$$

where a and x_0 are constants and also x_0 is the location of the center of the symmetrical wave at time $t = 0$. Note that the solitary wave moves to the right at a velocity $4a^2$ which is proportional to its amplitude $2a^2$, therefore taller waves travel faster than shorter ones. Zabusky and Kruskal [4] discovered numerically that these solitary wave solutions have the remarkable property that the interaction of two solitary wave solutions is elastic in their study of the continuum approximation to the nonlinear discrete mass string of Fermi-Pasta-Ulam. The critical observation was that the pulses seemed to retain their identities after each interaction. Because of their preservation of form through nonlinear interaction and their resemblance to particles, Zabusky and Kruskal [4] coined the name soliton for such waves.

2.2 Solitary Wave Interactions

It is well-known that long-time asymptotic behaviour of two dimensional unidirectional shallow water waves in the case of weak nonlinearity is described by the Korteweg-de Vries (KdV) equation [3]. Since, the inverse scattering transform (IST) for exactly solving the KdV equation was found by Gardner, Kruskal and Miura [27], the interesting features of the collision between solitary waves had been revealed: When two solitary waves approach closely, they interact, exchange their energies and position with one another, and, then separate off, regaining their original forms. Throughout the whole process of the collision, the solitary waves are remarkably stable entities preserving their identities through the interaction. The unique effect due to the collision is their phase shifts. It is believed that this striking colliding property of solitary waves can only be preserved in a conservative system.

The interactions of solitary waves can be classified as head-on collisions (the counterpropagating waves) and overtaking collisions (the copropagating waves). According to IST, all the KdV solitary waves travel in the same direction, under the boundary conditions vanishing at infinity [27, 28]; so for overtaking collision between solitary waves, one can use the IST to obtain the overtaking colliding effect of solitary waves. However, for the head-on collision between solitary waves, one must employ some kind of asymptotic expansion to solve the original field equations. In this regard, it is useful to briefly review the studies on the head-on collision between two solitary waves in the free surface of an inviscid homogeneous fluid lying over a horizontal bottom. Byatt-Smith [29] investigated the interaction between two weakly nonlinear solitary waves travelling in opposite directions and explicitly determined the maximum amplitude of the collision. When, in the absence of viscosity and surface tension, two waves of equal amplitude collide, one has the problem of solitary wave reflection from a vertical wall. For this case the maximum elevation of the wave at the wall, the run-up, exceeds twice the amplitude of the incident solitary wave. Oikawa and Yajima [30] explicitly computed the spatial phase shift incurred after collision by adopting a perturbation approach. Maxworthy [31] performed experiments in a wave tank investigating both endwall and wave-wave collisions. Results of the experiment show that run-ups are in qualitative agreement with [29]. However, his measured phase shifts were, within experimental error, independent of ϵ thus the dependence on ϵ was not verified with [30]. After that, Su and Mirie [6] carried out a perturbation analysis of two colliding solitary waves to third order approximation. They found that, the maximum amplitude during the collision agree very well with the reflexion experiments of Maxworthy [31]. They are consistently lower than the wave-wave experiment of Maxworthy. The total phase shifts represent a retardation of the waves during their collision. However, they do not seem able to account for the experimental result which measured amplitude-independent phase shifts. Each solitary wave sheds a secondary wave. These secondary waves propagate in the opposite direction of their parent waves. Their amplitudes decrease in time owing to dispersion. Maxworthy indicates appearance of the secondary wave in his reflexion experiment. For a comparison with the numerical results, we can refer Fenton and Rienecker [32], Cooker et al. [33] and Craig et al. [34]. As is stated by Cooker et al. [33], the numerical results on run-up are consistent with the predictions of Su and Mirie [6]. Both Cooker et al. [33] and Craig et al. [34] have the results correspond to the experimental observations in Maxworthy [31]. The existence of a residual is qualitatively consistent with the asymptotic predictions of Su and Mirie [6], however, on a quantitative level numerical data of Craig et al. [34] are at odds with their findings.

2.3 Water Waves

In fluid dynamics, the physical quantities, such as mass, velocity, energy, etc., are usually regarded as being spread continuously throughout the region of consideration; this is often termed the continuum assumption and continuum derivations are based on conservation principles. Now we focus on the results derived from conservation laws and, in particular, how they relate to water waves. We will use $\rho = \rho(\boldsymbol{x}^\star, t^\star)$ to denote the fluid mass density, $\boldsymbol{v} = \boldsymbol{v}(\boldsymbol{x}^\star, t^\star)$ the fluid velocity, P the pressure, $\boldsymbol{F}$ a given external force, and ν the kinematic viscosity that is due to frictional forces. In vector notation, the relevant equations of fluid dynamics we will consider are:

$$\begin{aligned}\frac{\partial \rho}{\partial t^\star} + \nabla\cdot(\rho\boldsymbol{v}) &= 0,\\ \rho\left(\frac{\partial \boldsymbol{v}}{\partial t^\star} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v}\right) &= \boldsymbol{F} - \boldsymbol{\nabla P} + \nu\Delta\boldsymbol{v},\end{aligned}$$

where the former is called conservation of mass and the latter is conservation of linear momentum. When $\rho = \rho_0$ is constant, the first equation then describes an incompressible fluid: $\boldsymbol{\nabla}\cdot\boldsymbol{v} = 0$, also called the divergence equation. The divergence and the momentum equations are often called the incompressible Navier-Stokes equations. We will consider the free surface water wave problem, which (interior to the fluid) is the inviscid reduction ($\nu = 0$) of the above equations; these equations are called the Euler equations.

For our discussion of water waves, we will use the incompressible Navier-Stokes description with constant density $\rho = \rho_0$ and we will assume an ideal fluid: that is, a fluid with zero viscosity. Thus, an ideal, incompressible fluid is described by the following Euler equations:

$$\boldsymbol{\nabla}\cdot\boldsymbol{v} = 0, \tag{2.6}$$

$$\frac{\partial \boldsymbol{v}}{\partial t^\star} + (\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v} = \frac{1}{\rho_0}(\boldsymbol{F} - \boldsymbol{\nabla}P). \tag{2.7}$$

Suppose now that the external force is conservative, that is, we can write $\boldsymbol{F} = -\boldsymbol{\nabla}U$, for some scalar potential U. We can then write the equation (2.7) as

$$\frac{\partial \boldsymbol{v}}{\partial t^\star} + (\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v} = -\boldsymbol{\nabla}\left(\frac{U+P}{\rho_0}\right).$$

Using the vector identity

$$(\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v} = \frac{1}{2}\boldsymbol{\nabla}(\boldsymbol{v}\cdot\boldsymbol{v}) - \boldsymbol{v}\times(\boldsymbol{\nabla}\times\boldsymbol{v}),$$

gives

$$\frac{\partial \boldsymbol{v}}{\partial t^\star} - \boldsymbol{v}\times(\boldsymbol{\nabla}\times\boldsymbol{v}) = -\boldsymbol{\nabla}\left(\frac{1}{2}\boldsymbol{v}\cdot\boldsymbol{v} + \frac{U+P}{\rho_0}\right). \tag{2.8}$$

Now define the vorticity vector as $\boldsymbol{\omega} = \nabla\times\boldsymbol{v}$, which is a local measure of the degree to which the fluid is spinning; more precisely, $\frac{1}{2}\|\nabla\times\boldsymbol{v}\|$ (note that $\|\boldsymbol{v}\|^2 = \boldsymbol{v}\cdot\boldsymbol{v}$) is the angular speed of an infinitesimal fluid element. Taking the curl of the last equation and noting that the curl of a gradient vanishes,

$$\frac{\partial \boldsymbol{\omega}}{\partial t^\star} - \boldsymbol{\nabla} \times (\boldsymbol{v} \times \boldsymbol{\omega}) = 0.$$

Finally, using the vector identity $\boldsymbol{\nabla} \times (\boldsymbol{F} \times \boldsymbol{G}) = (\boldsymbol{G} \cdot \boldsymbol{\nabla})\boldsymbol{F} - (\boldsymbol{F} \cdot \boldsymbol{\nabla})\boldsymbol{G} + (\boldsymbol{\nabla} \cdot \boldsymbol{G})\boldsymbol{F} - (\boldsymbol{\nabla} \cdot \boldsymbol{F})\boldsymbol{G}$ for vector functions $\boldsymbol{F}$ and $\boldsymbol{G}$ and recalling that the divergence of the curl vanishes, we arrive at the so-called vorticity equation:

$$\frac{\partial \boldsymbol{\omega}}{\partial t^\star} = (\boldsymbol{\omega} \cdot \boldsymbol{\nabla})\boldsymbol{v} - (\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{\omega} \tag{2.9}$$

or

$$\frac{D\boldsymbol{\omega}}{Dt^\star} = \boldsymbol{\omega} \cdot \boldsymbol{\nabla} \boldsymbol{v}, \tag{2.10}$$

where the notation

$$\frac{D}{Dt^\star} = \frac{\partial}{\partial t^\star} + (\boldsymbol{v} \cdot \boldsymbol{\nabla})$$

have been used to signify the so-called convective or material derivative that moves with the fluid particle. Hence $\boldsymbol{\omega} = 0$ is a solution; moreover, from (2.10), it can be proven that if the vorticity is initially zero, then (if the solution exists) it is zero for all times. Such a flow is called irrotational. Physically, in an ideal fluid there is no mechanism that will produce "local rotation" if the fluid is initially irrotational. Often it is a good approximation to assume that a fluid is irrotational, with viscosity effects occurring only in thin regions of the fluid flow called boundary layers. Since we will consider water waves and will assume that the flow is irrotational. In such circumstances, it is convenient to introduce a velocity potential $\boldsymbol{v} = \boldsymbol{\nabla}\psi^\star$. Notice that the vorticity equation (2.10) is trivially satisfied since

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla}\psi^\star) = 0.$$

The Euler equations inside the fluid region can now also be simplified as

$$\boldsymbol{\nabla} \cdot \boldsymbol{v} = \boldsymbol{\nabla} \cdot \boldsymbol{\nabla}\psi^\star = \Delta\psi^\star = 0, \tag{2.11}$$

which is Laplace's equation; it is to be satisfied internal to the fluid, $0 < z^\star < h^\star(x^\star, y^\star, t^\star)$, where we denote the height of the fluid free surface to be $h^\star(x^\star, y^\star, t^\star)$ and the fluid is supported by a horizontal plane at $z^\star = 0$.

Next we discuss the boundary conditions that lead to complications; i.e., an unknown free surface and nonlinearities. We assume a flat, impenetrable bottom at $z^\star = 0$, so that no fluid can flow through. This results in the condition

$$\frac{\partial \psi^\star}{\partial z^\star} = 0, \quad z^\star = 0, \tag{2.12}$$

that is, normal velocity vanishes at the bottom $z^\star = 0$. On the free surface $z^\star = h^\star(x^\star, y^\star, t^\star)$ there are two conditions. The first is obtained from (2.8). Using the fact that $\boldsymbol{\nabla}$ and $\dfrac{\partial}{\partial t}$ commute,

$$\boldsymbol{\nabla}\left(\frac{\partial \psi^\star}{\partial t^\star} + \frac{1}{2}\|\boldsymbol{v}\|^2 + \frac{U+P}{\rho_0}\right) = 0,$$

which gives

$$\frac{\partial \psi^\star}{\partial t^\star} + \frac{1}{2}\|\boldsymbol{v}\|^2 + \frac{U+P}{\rho_0} = f(t^\star), \tag{2.13}$$

where we recall $\|\boldsymbol{v}\|^2 = \psi^{\star 2}_{x^\star} + \psi^{\star 2}_{y^\star} + \psi^{\star 2}_{z^\star}$. Since the physical quantity is $\boldsymbol{v} = \boldsymbol{\nabla}\psi^\star$, we can add an arbitrary function of time (independent of space) to $\psi^\star$,

$$\psi^\star \to \psi^\star + \int_0^{t^\star} f(\tau)d\tau,$$

to get the so-called Bernoulli, dynamic, or pressure equation,

$$\frac{\partial \psi^\star}{\partial t^\star} + \frac{1}{2}\|\boldsymbol{v}\|^2 + \frac{U+P}{\rho_0} = 0. \tag{2.14}$$

From now on, we will neglect surface tension and assume that the dominant force is the buoyancy force, $\boldsymbol{F} = -\boldsymbol{\nabla}(\rho_0 g z^\star)$, which implies that $U = \rho_0 g z^\star$, where g is the gravitational constant of acceleration. Also, we take the pressure to vanish (i.e., $P = 0$) on the free surface, yielding:

$$\frac{\partial \psi^\star}{\partial t^\star} + \frac{1}{2}\|\boldsymbol{\nabla}\psi^\star\|^2 + g h^\star = 0, \quad z^\star = h^\star(x^\star, y^\star, t^\star), \tag{2.15}$$

on the free surface.

We consider the case of a body of water with air above it and let the interface be described by

$$F(x^\star, y^\star, z^\star, t^\star) = 0. \tag{2.16}$$

The second equation governing the free surface is derived from the assumption that the interface is defined by the property that fluid does not cross it. Hence the velocity of the fluid normal to the interface must be equal to the velocity of the interface normal to itself. The normal velocity of a surface defined by (2.16) is

$$\frac{-F_{t^\star}}{\sqrt{F_{x^\star}^2 + F_{y^\star}^2 + F_{z^\star}^2}}.$$

The normal velocity of the fluid is

$$\frac{\boldsymbol{v} \cdot \boldsymbol{\nabla} F}{\sqrt{F_{x^\star}^2 + F_{y^\star}^2 + F_{z^\star}^2}}.$$

The condition that these be equal is

$$\frac{DF}{Dt^\star} = \frac{\partial F}{\partial t^\star} + \boldsymbol{v} \cdot \boldsymbol{\nabla} F = 0. \tag{2.17}$$

This shows that if a fluid packet is initially on the free surface, then it will stay there. It is convenient to describe the surface by $z^\star = h^\star(x^\star, y^\star, t^\star)$ and choose $F = z^\star - h^\star(x^\star, y^\star, t^\star) = 0$ in (2.17). Then

$$\frac{Dz^\star}{Dt^\star} = \frac{Dh^\star}{Dt^\star}$$

implies

$$\frac{\partial \psi^\star}{\partial z^\star} = \frac{\partial h}{\partial t^\star} + \boldsymbol{v} \cdot \boldsymbol{\nabla} h^\star, \quad z^\star = h^\star(x^\star, y^\star, t^\star), \tag{2.18}$$

on the free surface where we have used $\boldsymbol{v} = \left(\frac{Dx^\star}{Dt^\star}, \frac{Dy^\star}{Dt^\star}, \frac{Dz^\star}{Dt^\star} \right)$. Equation (2.18) is often referred to as the kinematic condition (For a more detailed discussion, see [35]).

2.4 Fluid-filled Elastic Tubes

The propagation of linear and nonlinear waves in fluid-filled elastic tubes is a problem of interest since the time of Thomas Young [36]. The measurement [37] for the simultaneous changes in amplitudes and form of the flow and pressure waves at five sites from the ascending aorta to the saphenous artery in dog shown that the pulsatile character of the blood wave is soliton-like and it suggests a possible interpretation in terms of solitons. The blood flow in arteries can be considered as an incompressible fluid flowing in a thin non-linear elastic tube. Theoretical investigations for the blood waves by weakly nonlinear theory have been developed by [38–41]. It is shown that the dynamics of the blood waves are governed by the KdV or modified KdV equations. The solitary wave model gives a reasonable explanation for the peaking and steepening of pulsatile waves in arteries. Head-on collision of solitary waves in fluid-filled elastic tubes (a model for arteries) had been studied by several researchers [7, 42, 43], in all of which the method proposed by Su and Mirie [6] have been employed.

To study the head-on collision of the blood solitary waves, we assume that the blood waves propagate in a one- dimensional elastic tube, which is deemed to be a model for large artery, filled with an incompressible inviscid fluid, which is considered to be a simple model for blood. We also assume that the arteries are circularly cylindrical homogeneous tube with non-linear elasticity. Then, the equations governing the conservation of mass and of the balance of linear momentum in the axial direction may be given as follows, respectively [44],

$$\frac{\partial A}{\partial t^\star} + \frac{\partial}{\partial x^\star}\left(A v^\star\right) = 0, \tag{2.19}$$

$$\frac{\partial v^\star}{\partial t^\star} + v^\star \frac{\partial v^\star}{\partial x^\star} + \frac{1}{\rho}\frac{\partial P^\star}{\partial x^\star} = 0, \tag{2.20}$$

where ρ is the density of fluid, $A(x^\star, t^\star)$ denotes the cross-sectional area of the tube, $v^\star$ the axial velocity of the fluid and $P^\star$ the pressure of the fluid.

Equations (2.19) and (2.20) give only two relations to determine the unknown functions A, $v^\star$ and $P^\star$. In order to have a complete determination of these field variables, a third equation describing the radial motion of the wall

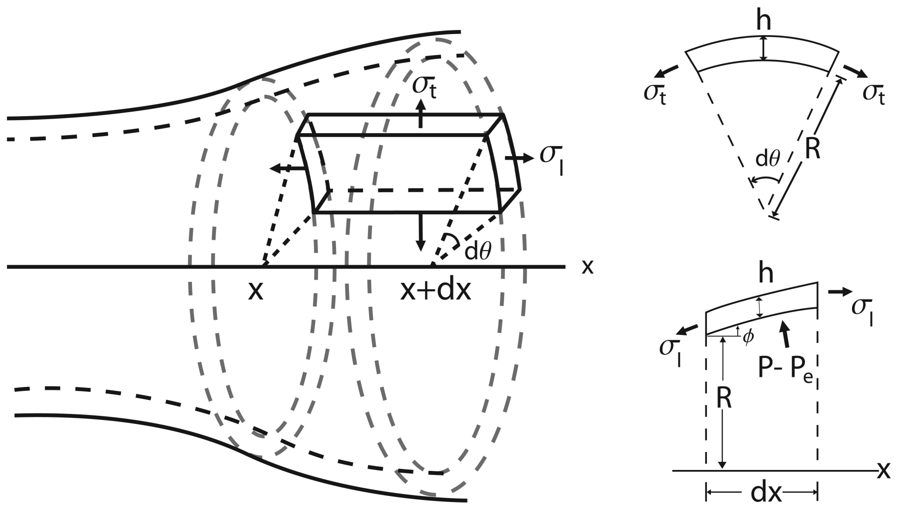

Figure 2.1: Small segment of the tube wall and forces acting on this segment.

under the forces exerted by the fluid is necessary (These forces are shown in Figure 2.1). Thus, by using the assumptions given in [45], this equation can be given as follows:

$$\frac{\rho_0 H}{2\pi R_0}\frac{\partial^2 S^\star}{\partial t^{\star 2}} = (P^\star - P_e) - \frac{Eh}{2R_0}\frac{S^\star}{\pi R_0^2}\frac{\left(1+\alpha\frac{S^\star}{2\pi R_0^2}\right)}{\left(1+\frac{S^\star}{2\pi R_0^2}\right)} \tag{2.21}$$

where ρ_0 denotes the constant mass density of the wall of the tube, R_0 equilibrium radius of the tube, H effective inertial thickness, h thickness of the wall in which the material participates in the elastic deformation, P_e the pressure outside the tube which can be regarded as same as the atmospheric pressure, E the Young's modulus, α the nonlinear coefficient of elasticity and $S^\star = A - A_0$ the change in the cross-sectional area of the tube. Here, it was assumed that the variable $S^\star \approx 2\pi R_0 u_r$ where $u_r = R - R_0$ is the radial displacement. The incompressibility of the wall and the tissue gives the following equations

$$RH = R_0 H_0, \qquad Rh = R_0 h_0, \tag{2.22}$$

in which H_0 and h_0 denote the equilibrium values of effective inertial thickness and the thickness of the wall, respectively. Now it is convenient to introduce the following non-dimensional quantities

$$x^\star = X_0 x, \quad t^\star = T_0 t, \quad v^\star = \frac{X_0}{T_0} v, \quad P^\star - P_e = p_0 P, \quad S^\star = \pi R_0^2 S, \qquad (2.23)$$

where

$$X_0 = \left(\frac{\rho_0 R_0 H_0}{2\rho}\right)^{1/2}, \quad T_0 = \left(\frac{\rho_0 R_0^2 H_0}{E h_0}\right)^{1/2}, \quad p_0 = \frac{E h_0}{2 R_0}.$$

Introducing (2.22) and (2.23) into the equations (2.19)-(2.21), the following non-dimensional equations are obtained

$$\frac{\partial S}{\partial t} + \frac{\partial v}{\partial x} + \frac{\partial}{\partial x}(Sv) = 0, \qquad (2.24)$$

$$\frac{\partial v}{\partial t} + \frac{\partial P}{\partial x} + \frac{\partial}{\partial x}\left(\frac{v^2}{2}\right) = 0, \qquad (2.25)$$

$$P = \frac{2}{2+S}\frac{\partial^2 S}{\partial t^2} + \frac{2S(2+\alpha S)}{(2+S)^2}. \qquad (2.26)$$

A more detailed discussion can be found in [44–46].

2.5 Perturbation Methods

If one wants to study physical problems then the general equations obtained from first principles (like the Euler or Navier-Stokes equations governing fluid motion on a free surface mentioned in the previous section) are too difficult to handle using linear methods or, in most situations, by direct numerical simulation. Therefore, before we can study the solutions of the governing equations we must first obtain useful and manageable equations. For that we need to simplify the general equations, while retaining the essential phenomena we want to study. For this purpose, we introduce, first, the reductive perturbation method and then Poincaré-Lighthill-Kuo (PLK) method.

2.5.1 Reductive Perturbation Method

In the study of the asymptotic behaviour of nonlinear dispersive waves, Gardner and Morikawa [12] introduced the scale transformation

$$\begin{aligned} \xi &= \epsilon^{\alpha}(x - \lambda t), \\ \tau &= \epsilon^{\beta} t. \end{aligned}$$

This scale transformation, called the Gardner-Morikawa transformation, may be derived from the linearized asymptotic behaviour of long waves. Gardner and Morikawa combined this transformation with a perturbation expansion of the dependent variable so as to describe the nonlinear asymptotic behaviour and they arrived the KdV equation as a single tractable equation describing the asymptotic behaviour of a wave. The reductive perturbation method has been developed and formulated in a general way by Taniuti and Wei [22], Taniuti and Washimi [47], Taniuti and Yajima [48] and Taniuti [49].

2.5.2 The long wave approximation

The reductive perturbation method for long waves was established by Taniuti and Wei [22]. This method is applicable to both dispersive and dissipative systems governed by the system of equations given as:

$$\frac{\partial \boldsymbol{U}}{\partial t} + \boldsymbol{A}\frac{\partial \boldsymbol{U}}{\partial x} + \sum_{\beta=1}^{s}\prod_{\alpha=1}^{p}(\boldsymbol{H}_{\alpha}^{\beta}\frac{\partial}{\partial t} + \boldsymbol{K}_{\alpha}^{\beta}\frac{\partial}{\partial x})\boldsymbol{U} = 0, \quad p \geq 2, \tag{2.27}$$

where $\boldsymbol{U}$ is a column vector with the n components $u_1, u_2, ..., u_n$. Here $\boldsymbol{A}, \boldsymbol{H}_{\alpha}^{\beta}$ and $\boldsymbol{K}_{\alpha}^{\beta}$ are $n \times n$ matrices, all of which are functions of $\boldsymbol{U}$. Introducing the Gardner-Morikawa transformation

$$\begin{aligned} \xi &= \epsilon^{a}(x - \lambda t), \\ \tau &= \epsilon^{a+1} t, \\ a &= \frac{1}{p-1}, \end{aligned} \tag{2.28}$$

we shall assume expansions around a constant solution $\boldsymbol{U}_0$ of the form

$$\boldsymbol{U} = \sum_{j=1}^{\infty} \epsilon^j \boldsymbol{U}_j, \;\; \boldsymbol{A} = \sum_{j=0}^{\infty} \epsilon^j \boldsymbol{A}_j, \;\; \boldsymbol{H}_\alpha^\beta = \sum_{j=0}^{\infty} \epsilon^j \boldsymbol{H}_{\alpha j}^\beta, \;\; \boldsymbol{K}_\alpha^\beta = \sum_{j=0}^{\infty} \epsilon^j \boldsymbol{K}_{\alpha j}^\beta. \tag{2.29}$$

Substituting (2.29) into (2.27) and using (2.28) enables us to rewrite (2.27) in terms of derivatives with respect to ξ and τ. Equating the coefficients of like powers in ϵ to zero we obtain

$$O(\epsilon^{\alpha+1}) : \qquad (\boldsymbol{A}_0 - \lambda \boldsymbol{I}) \frac{\partial \boldsymbol{U}_1}{\partial \xi} = 0, \tag{2.30}$$

$$O(\epsilon^{\alpha+2}) : \qquad (\boldsymbol{A}_0 - \lambda \boldsymbol{I}) \frac{\partial \boldsymbol{U}_2}{\partial \xi} + \frac{\partial \boldsymbol{U}_1}{\partial \tau} + [\boldsymbol{U}_1 \cdot (\nabla_{\boldsymbol{u}} \boldsymbol{A})_0] \frac{\partial \boldsymbol{U}_1}{\partial \xi}$$
$$+ \sum_{\beta=1}^{s} \prod_{\alpha=1}^{p} (-\lambda \boldsymbol{H}_{\alpha 0}^\beta + \boldsymbol{K}_{\alpha 0}^\beta) \frac{\partial^p \boldsymbol{U}_1}{\partial \xi^p} = 0, \tag{2.31}$$

where $\nabla_{\boldsymbol{u}}$ denotes the gradient operator with respect to $\boldsymbol{U}$, and $\boldsymbol{U} \cdot \nabla_{\boldsymbol{u}}$ represents the operator $\Sigma_{i=1}^n u_i (\partial / \partial u_i)$ and $\boldsymbol{A}_1$ is written as $\boldsymbol{U}_1 \cdot (\nabla_{\boldsymbol{u}} \boldsymbol{A})_0$. Introducing the right eigenvector $\boldsymbol{R}$ of $\boldsymbol{A}_0$ corresponding to λ, so that

$$(\boldsymbol{A}_0 - \lambda \boldsymbol{I}) \boldsymbol{R} = 0, \tag{2.32}$$

we can solve (2.30) in the form

$$\boldsymbol{U}_1 = \phi_1(\xi, \tau) \boldsymbol{R} + \boldsymbol{V}_1(\tau) \tag{2.33}$$

where ϕ_1 is one of the components of $\boldsymbol{U}_1$. Here $\boldsymbol{V}_1$ is an arbitrary vector valued function of τ to be determined according to an appropriate boundary condition for $\boldsymbol{U}_1$. In order that (2.31) is solvable algebraically for $\partial \boldsymbol{U}_2 / \partial \xi$, there must exist a compatibility condition. To obtain this condition we multiply equation (2.31) on the left by the left eigenvector $\boldsymbol{L}$, when we get

$$\boldsymbol{L} \cdot \frac{\partial \boldsymbol{U}_1}{\partial \tau} + \boldsymbol{L} \cdot [\boldsymbol{U}_1 \cdot (\nabla_{\boldsymbol{U}} \boldsymbol{A})_0 \frac{\partial \boldsymbol{U}_1}{\partial \xi}) + \boldsymbol{L} \cdot \sum_{\beta=1}^{s} \prod_{\alpha=1}^{p} (-\lambda \boldsymbol{H}_{\alpha 0}^\beta + \boldsymbol{K}_{\alpha 0}^\beta) \frac{\partial^p \boldsymbol{U}_1}{\partial \xi^p} = 0. \tag{2.34}$$

When the boundary condition for $\boldsymbol{U}_1$ is such that $\boldsymbol{U} \to \boldsymbol{U}_0$ as $x \to \infty$, so that $\boldsymbol{U}_i \to 0$ as $x \to \infty$, $(i \geq 1)$, we may set $\boldsymbol{V}_1$ equal to zero in (2.33). Using (2.33) with $\boldsymbol{V}_1 = 0$ in (2.34), we get

$$\frac{\partial \phi_1}{\partial \tau} + c_1 \phi_1 \frac{\partial \phi_1}{\partial \xi} + c_2 \frac{\partial^p \phi_1}{\partial \xi^p} = 0. \tag{2.35}$$

where the constants c_1 and c_2 are given by

$$\begin{aligned} c_1 &= \boldsymbol{L} \cdot [\boldsymbol{R} \cdot (\nabla_{\boldsymbol{U}} \boldsymbol{A})_0 \boldsymbol{R}]/(\boldsymbol{L} \cdot \boldsymbol{R}), \\ c_2 &= \boldsymbol{L} \cdot \sum_{\beta=1}^{s} \prod_{\alpha=1}^{p} (-\lambda \boldsymbol{H}_{\alpha 0}^{\beta} + \boldsymbol{K}_{\alpha 0}^{\beta}) \boldsymbol{R}/(\boldsymbol{L} \cdot \boldsymbol{R}). \end{aligned} \tag{2.36}$$

When $p = 3$, equation (2.35) is the KdV equation, whereas for $p = 2$, equation (2.35) is Burgers' equation for the one dimensional flow of a compressible viscous fluid.

2.5.3 Poincaré-Lighthill-Kuo (PLK) Method

This method goes back to the nineteenth century when astronomers, such as Lindstedt [50], Bohlin [51] and Gyldén [52] devised techniques to avoid the appearance of secular terms in perturbation solutions of differential equations. Poincaré [53] devised a method for finding the periodic solution of a system of first order equations

$$\frac{dX_i}{dt} = X_i(x_1, x_2, \ldots, x_n; \epsilon), \qquad (i = 1, 2, \ldots, n), \tag{2.37}$$

where t is the time variable and ϵ is a small parameter representing the perturbation influences. The equations with $\epsilon = 0$, corresponding to the unperturbed system, are particularly simple, and a periodic solution with period T_0 can be easily found. The essence of Poincaré's method is the expansion of the perturbed solution in the parameter ϵ. Not only the variables

$$x_i = x_i^{(0)} + \epsilon x_i^{(1)} + \epsilon^2 x_i^{(2)} + \ldots$$

are expanded, but also the period T

$$T = T_0 + \epsilon T_1 + \epsilon^2 T_2 + \dots .$$

However, for nearly sixty years no extension of the principle of this method was made, and the full potentiality of Poincaré's invention remained unexploited. Lighthill [54] developed a technique for rendering approximate solutions to physical problems uniformly valid and introduced a very important extension of Poincaré's method. Lighthill's objective was to improve the well-known method of perturbation for calculating the approximate solution of a physical problem. The perturbation method is based upon the concept of expanding the exact solution in a power series of the small parameter ϵ, the zeroth order solution being independent of ϵ, the first order solution proportional to ϵ, etc. This method, elementary in principle and straightforward in execution, is very effective and yields useful results for a large class of problems. Nevertheless there are problems, not at all infrequent, where the zeroth order solution contains a singularity at a point or on a line within the domain of interest. Then not only will the singularity again appear at the same location in the higher order solutions, but it will become progressively more severe as the order of the solution increases. The power series expansion in ϵ breaks down near such singularities, and the classical perturbation fails to give a usable solution near the singular points.

Lighthill's method is designed to eliminate such difficulties and to render the expansion uniformly valid, or of uniform accuracy, over the whole domain of interest. The principle is to expand not only the dependent variable u, but also the independent variables x and y in power series of ϵ. Then

$$u = u_0(\xi, \eta) + \epsilon u_1(\xi, \eta) + \epsilon^2 u_2(\xi, \eta) + \dots , \tag{2.38}$$

$$x = \xi + \epsilon x_1(\xi, \eta) + \epsilon^2 x_2(\xi, \eta) + \dots , \tag{2.39}$$

$$y = \eta + \epsilon y_1(\xi, \eta) + \epsilon^2 y_2(\xi, \eta) + \dots , \tag{2.40}$$

where ξ, η take the place of the original independent variables x, y. If we neglect the higher order terms in u of (2.38), then the approximate solution is simply the zeroth order perturbation solution with the coordinates stretched or distorted by the transformations (2.39) and (2.40). This fact has led several authors to call Lighthill's method, the method of coordinate perturbation or method of strained coordinates.

Lighthill applied his method to problems involving partial differential equations when the zeroth order solution is obtained from a reduced linear equation of equal order as the exact equation. It soon becomes apparent, however, that Lighthill's original purpose of uniform validity throughout the domain of interest cannot always be realized. In many problems a good zeroth order approximation can be obtained only if a "boundary layer" solution is used. Kuo [55] first recognized this necessity in his solution of the problem of the laminar incompressible boundary layer on a flat plate and extended the Poincaré's original concept.

To illustrate the principle of the PLK method, let us consider the following first order ordinary differential equation

$$(x + \epsilon u)\frac{du}{dx} + u = 0, \tag{2.41}$$

which can also be written as

$$\frac{d}{dx}\left(xu + \epsilon\frac{u^2}{2}\right) = 0.$$

Then by integration, we obtain

$$xu + \epsilon\frac{u^2}{2} = C_0. \tag{2.42}$$

If we impose the boundary condition $u(1) = 1$, the exact solution of the equation (2.41) is

$$u = -\frac{x}{\epsilon} + \sqrt{\left(\frac{x}{\epsilon}\right)^2 + \frac{2}{\epsilon} + 1}. \tag{2.43}$$

Now let us apply the classical perturbation method for the equation (2.41), i.e., expand u in powers of ϵ

$$u(x) = u_0(x) + \epsilon u_1(x) + \epsilon^2 u_2(x) + \ldots. \tag{2.44}$$

Substituting (2.44) into (2.41), and then equating the like powers of ϵ, we have

$$\mathcal{O}(\epsilon^0): \qquad x\frac{du_0}{dx} + u_0 = 0, \tag{2.45}$$

$$\mathcal{O}(\epsilon): \qquad x\frac{du_1}{dx} + u_1 = -u_0\frac{du_0}{dx}, \tag{2.46}$$

$$\mathcal{O}(\epsilon^2): \qquad x\frac{du_2}{dx} + u_2 = -u_0\frac{du_1}{dx} - u_1\frac{du_0}{dx}. \tag{2.47}$$

Then the solution of the $\mathcal{O}(\epsilon^0)$ equation with the boundary condition $u(1) = 1$ yields

$$u_0(x) = \frac{1}{x}. \tag{2.48}$$

Substituting (2.48) into (2.46), we obtain the solution of the $\mathcal{O}(\epsilon)$ equation as

$$u_1(x) = -\frac{1}{2x^3} + \frac{C_1}{x}.$$

But now the boundary condition requires $u_1(1) = 0$ which gives $C_1 = \frac{1}{2}$. Then we have

$$u_1(x) = \frac{1}{2x}\left(1 - \frac{1}{x^2}\right). \tag{2.49}$$

Similarly, for the solution of the equation (2.47), we obtain

$$u_2(x) = \frac{1}{2x}\left(1 - \frac{1}{x^2}\right) - \frac{1}{2x}\left(1 - \frac{1}{x^4}\right). \tag{2.50}$$

The function $u_0(x)$ has a singularity at $x = 0$, and (2.49) and (2.50) show that this singularity becomes worse as the order of the perturbation solution is increased. The solution so obtained is thus worthless near $x = 0$.

Now let us apply a different procedure, expand both u and x in powers of ϵ as required by the PLK method:

$$\begin{aligned} u &= u_0(\xi) + \epsilon u_1(\xi) + \ldots, \\ x &= \xi + \epsilon x_1(\xi) + \ldots. \end{aligned} \tag{2.51}$$

The differential equation (2.41) can now be written as

$$(x + \epsilon u)\frac{du}{d\xi} + u\frac{dx}{d\xi} = 0, \tag{2.52}$$

where

$$\frac{dx}{d\xi} = 1 + \epsilon x_1'(\xi) + \ldots.$$

By introducing (2.51) into (2.52) and equating the like powers of ϵ, we obtain

$$\mathcal{O}(\epsilon^0): \qquad \xi\frac{du_0}{d\xi} + u_0 = 0, \tag{2.53}$$

$$\mathcal{O}(\epsilon): \qquad \xi\frac{du_1}{d\xi} + u_1 = -(x_1 + u_0)\frac{du_0}{d\xi} - u_0\frac{dx_1}{d\xi}. \tag{2.54}$$

Now the solution of (2.53) gives

$$u_0(\xi) = \frac{C_2}{\xi}.$$

If we impose the condition $x_1(1) = 0$ such that $x = 1$ for $\xi = 1$, then $u_0(1) = 1$ is required by the boundary condition $u(1) = 1$. Thus

$$u_0(\xi) = \frac{1}{\xi}. \tag{2.55}$$

By introducing (2.55) into (2.54), we obtain

$$\frac{d}{d\xi}(\xi u_1) = -\frac{1}{\xi}\frac{dx_1}{d\xi} + \frac{1}{\xi^2}x_1 + \frac{1}{\xi^3}. \tag{2.56}$$

Now, if we ignore $x_1(\xi)$, the equation (2.56) yields the same solution as (2.49), in which the singularity becomes worse than the lower order solution as mentioned before. In order to avoid this, we take advantage of the additional freedom in the choice of x_1 by setting

$$\frac{1}{\xi}\frac{dx_1}{d\xi} - \frac{1}{\xi^2}x_1 = \frac{1}{\xi^3}. \tag{2.57}$$

The solution of (2.57) with the boundary condition $x_1(1) = 0$ is

$$x_1(\xi) = \frac{\xi}{2}\left(1 - \frac{1}{\xi^2}\right). \tag{2.58}$$

Now with $x_1(\xi)$ so determined, the solution of the equation (2.56) with the boundary condition $u_1(1) = 0$ yields $u_1 \equiv 0$. Then, up to this order of approximation, we have

$$u = \frac{1}{\xi},$$

$$x = \xi + \epsilon \frac{\xi}{2}\left(1 - \frac{1}{\xi^2}\right). \tag{2.59}$$

Now the interesting fact is that by eliminating ξ from the pair (2.59), we have exactly the solution for u as given by (2.43). Therefore in this case the PLK method not only removes the difficulty of the singularity at $x = 0$, but yields a solution which is, in fact, the exact solution. More detailed discussions about the PLK method can be found in Tsien [56] and Nayfeh [57].

Chapter 3

Re-examining the Head-on Collision Problem Between Two Solitary Waves in Shallow Water

3.1 Introduction

It is well known that one of the striking properties of solitons is their asymptotic preservation of form when they undergo a collision, as first remarked by Zabusky and Kruskal [4]. The unique effect due to collision is their phase shift. In a one-dimensional system, there are two distinct soliton interactions. One is the overtaking collision and the other is the head-on collision. Because of the multisoliton solutions of the Korteweg-de Vries (KdV) equation, the overtaking collision of solitary waves can be studied by the inverse scattering transformation method [27, 28] and Zou and Su [58]. For the numerical analysis of overtaking collisions of solitary waves it is worth of mentioning the works by Li and Sattinger [59] and Haragus et al. [60]. However, for the head-on collision between two solitary waves, one must examine the solitary waves propagating in opposite directions, and hence we need to employ a suitable asymptotic expansion to solve the original conducting fluid equations.

There have been several attempts to resolve the head-on collision problems in various media (see, for instance [29–31, 61, 62]). In this regard a fundamental approach for the study of head-on collision problems had been laid down by Su and Mirie [6], in which the Poincaré-Lighthill-Kuo (PLK) method had been employed for the asymptotic analysis of such collision problems. The PLK method is just the combination of the classical reductive perturbation method [22] with the strained coordinates. Su and Mirie [6] introduced the following stretched coordinates

$$\epsilon^{\frac{1}{2}}k(x - C_R t) = \xi - \epsilon k\theta(\xi, \eta),$$
$$\epsilon^{\frac{1}{2}}l(x + C_L t) = \eta - \epsilon l\phi(\xi, \eta), \tag{3.1}$$

where ϵ is the smallness parameter, C_R and C_L are the speeds of right and left going waves, k and l are the wave numbers of the right and left going waves, respectively, $\theta(\xi, \eta)$ and $\phi(\xi, \eta)$ are the trajectory functions which are to be determined from the requirements of non-secular solutions of the field variables. By expanding the field variables and the unknown trajectory functions $\theta(\xi, \eta)$ and $\phi(\xi, \eta)$ into asymptotic series in ϵ, they obtained a set of partial differential equations. They tried to obtain solutions to these evolution equations under the restrictions of non-secularity of the solution.

In their derivation Su and Mirie [6] made a statement, which is the most attractive point of their analysis, that "*although this term does not cause any secularity at this order but it will cause the secularity at higher order expansion, therefore, that term must vanish*". Assuming that this statement is correct, several researchers (see, for instance [42, 43, 63–70]) studied the head-on collision problems in various physical media and published them in various respected journals. But our calculations for higher order expansion showed that the term mentioned in their work does not cause any secularity in the solution; it rather occurs in the next order equation. This means the order of trajectory functions should be ϵ^2, not ϵ.

In this chapter, based on the above argument, we have studied the head-on collision of two solitary waves propagating in the shallow water by introducing a set of stretched coordinates in which the trajectory functions are of order of ϵ^2. Taking the non-dimensional form of the field equations used by Su and Mirie [6] and expanding the field variables and trajectory functions into power series of ϵ we obtained a set of differential equations governing the various terms in the perturbation expansion. By solving these equations under the non-secularity conditions we obtained the evolution equations which give the solitary wave solutions for both right and left going waves. Moreover, by deriving non-secular solutions for ϵ^3 order equations we obtained some restrictions which makes it possible to determine the trajectory functions of order ϵ^2. Using the conventional definition of phase shifts we determined the expressions of phase shifts of right and left going waves. As opposed to the results of previous studies our calculation shows that the phase shifts depend on both amplitudes of colliding waves and they are order of ϵ^2.

3.2 Basic Equations

We consider a plane irrotational flow of an incompressible fluid. Let $\psi^\star(x^\star, y^\star, t^\star)$ be the velocity potential related to the velocity components $u^\star$ and $v^\star$ in the $x^\star$ and $y^\star$ directions, respectively, by

$$u^\star = \frac{\partial \psi^\star}{\partial x^\star}, \qquad v^\star = \frac{\partial \psi^\star}{\partial y^\star}. \tag{3.2}$$

Then the Laplace equation (2.11) reads

$$\frac{\partial^2 \psi^\star}{\partial x^{\star 2}} + \frac{\partial^2 \psi^\star}{\partial y^{\star 2}} = 0. \tag{3.3}$$

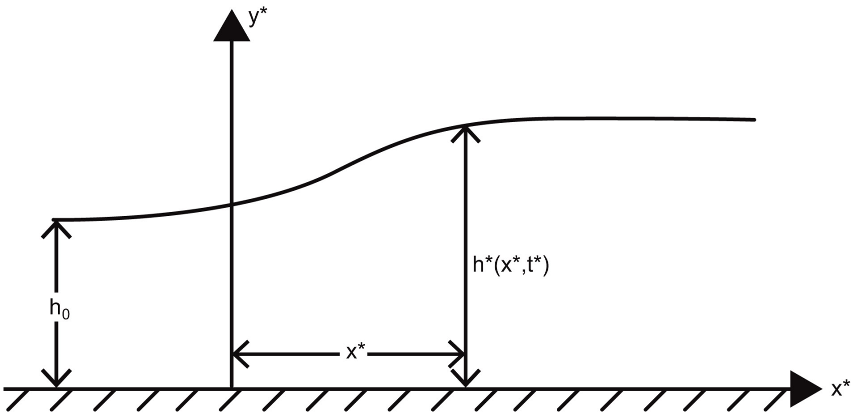

Figure 3.1: The geometry of the shallow water wave problem

The boundary conditions (2.12), (2.15) and (2.18) become

$$\frac{\partial \psi^\star}{\partial y^\star} = 0 \quad \text{at } y^\star = 0,$$
$$\frac{\partial h^\star}{\partial t^\star} + \frac{\partial \psi^\star}{\partial x^\star}\frac{\partial h^\star}{\partial x^\star} - \frac{\partial \psi^\star}{\partial y^\star} = 0 \quad \text{on } y^\star = h^\star,$$

$$\frac{\partial \psi^\star}{\partial t^\star} + \frac{1}{2}\left[\left(\frac{\partial \psi^\star}{\partial x^\star}\right)^2 + \left(\frac{\partial \psi^\star}{\partial y^\star}\right)^2\right] + g(h^\star - h_0) = 0 \quad \text{on } y^\star = h^\star, \tag{3.4}$$

where g is gravity acceleration of the earth. At this stage it is convenient to introduce the following non-dimensional quantities

$$\begin{aligned} &x^\star = h_0 x, \quad y^\star = h_0 y, \quad t^\star = \left(\frac{h_0}{g}\right)^{1/2} t, \\ &h^\star = h_0(1+\zeta), \quad \psi^\star = \left(g h_0^3\right)^{1/2} \psi, \end{aligned} \tag{3.5}$$

where h_0 is the still water level from the horizontal bottom. Introducing (3.5) into (3.3)-(3.4), the following non-dimensional equations are obtained

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0, \tag{3.6}$$

$$\begin{aligned} &\frac{\partial \psi}{\partial y} = 0 \quad \text{at } y = 0, \\ &\frac{\partial \zeta}{\partial t} + \frac{\partial \psi}{\partial x}\frac{\partial \zeta}{\partial x} - \frac{\partial \psi}{\partial y} = 0 \quad \text{at } y = 1+\zeta, \\ &\frac{\partial \psi}{\partial t} + \frac{1}{2}\left[\left(\frac{\partial \psi}{\partial x}\right)^2 + \left(\frac{\partial \psi}{\partial y}\right)^2\right] + \zeta = 0 \quad \text{at } y = 1+\zeta. \end{aligned} \tag{3.7}$$

Here we seek a power series solution for ψ of the form

$$\psi = \sum_{n=0}^{\infty} a_n(x,t) y^{2n}. \tag{3.8}$$

Introducing (3.8) into the Laplace equation (3.6) we obtain

$$a_1 = -\frac{1}{2!}\frac{\partial^2 a_0}{\partial x^2}, \quad a_2 = \frac{1}{4!}\frac{\partial^4 a_0}{\partial x^4}, \ldots. \tag{3.9}$$

Denoting the value of $\psi(x,y,t)$ at $y=0$ by $\Psi(x,t)$, the solution (3.8) can be written as follows

$$\psi = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\frac{\partial^{2n}\Psi}{\partial x^{2n}} y^{2n}. \tag{3.10}$$

The solution (3.10) also satisfies the boundary condition at $y=0$. Using the other boundary conditions we obtain

$$\frac{\partial\zeta}{\partial t}+\frac{\partial}{\partial x}\left\{(1+\zeta)w+\sum_{n=1}^{\infty}(-1)^n\frac{(1+\zeta)^{2n+1}}{(2n+1)!}\frac{\partial^{2n}w}{\partial x^{2n}}\right\}=0, \tag{3.11}$$

$$\frac{\partial w}{\partial t}+\frac{\partial}{\partial x}\left\{\zeta+\frac{w^2}{2}+\sum_{n=1}^{\infty}(-1)^n\frac{(1+\zeta)^{2n}}{(2n)!}\left[\frac{\partial^{2n}w}{\partial t\partial x^{2n-1}}\right.\right.$$

$$\left.\left.+\frac{1}{2}\sum_{m=0}^{2n}(-1)^m\binom{2n}{m}\frac{\partial^m w}{\partial x^m}\frac{\partial^{2n-m}w}{\partial x^{2n-m}}\right]\right\}=0, \tag{3.12}$$

where $w=\dfrac{\partial\Psi}{\partial x}$ and $\dbinom{2n}{m}$ is the binomial coefficient.

3.3 PLK Method

As stated in the Introduction section, following Su and Mirie [6], we introduce the following stretched coordinates

$$\epsilon^{\frac{1}{2}}k(x-C_Rt)=\xi-\epsilon k\theta(\xi,\eta), \tag{3.13}$$

$$\epsilon^{\frac{1}{2}}l(x+C_Lt)=\eta-\epsilon l\phi(\xi,\eta), \tag{3.14}$$

where ϵ is the smallness parameter representing the order of nonlinearity, k and l are the dimensionless wave numbers of order unity for the right and left going waves, respectively, and C_R and C_L, are the speeds of right and left going waves, $\theta(\xi,\eta)$ and $\phi(\xi,\eta)$ are two unknown functions to be determined from the solution. Then, the following differential operators can be introduced:

$$\frac{\partial}{\partial t}+C_R\frac{\partial}{\partial x}=\frac{\epsilon^{\frac{1}{2}}}{D}(C_R+C_L)\left[l\frac{\partial}{\partial\eta}+\epsilon kl\left(\frac{\partial\theta}{\partial\eta}\frac{\partial}{\partial\xi}-\frac{\partial\theta}{\partial\xi}\frac{\partial}{\partial\eta}\right)\right], \tag{3.15}$$

$$\frac{\partial}{\partial t}-C_L\frac{\partial}{\partial x}=-\frac{\epsilon^{\frac{1}{2}}}{D}(C_R+C_L)\left[k\frac{\partial}{\partial\xi}+\epsilon kl\left(\frac{\partial\phi}{\partial\xi}\frac{\partial}{\partial\eta}-\frac{\partial\phi}{\partial\eta}\frac{\partial}{\partial\xi}\right)\right], \tag{3.16}$$

where

$$D=\left(1-\epsilon k\frac{\partial\theta}{\partial\xi}\right)\left(1-\epsilon l\frac{\partial\phi}{\partial\eta}\right)-\epsilon^2kl\frac{\partial\theta}{\partial\eta}\frac{\partial\phi}{\partial\xi}. \tag{3.17}$$

Introducing (3.15) and (3.16) into (3.11) and (3.12) we obtain

$$\left[\frac{\partial}{\partial t} \pm C_{R,L}\frac{\partial}{\partial x}\right][w \pm \zeta] + \frac{\partial}{\partial x}F_{\pm} = 0, \tag{3.18}$$

where $F_{\pm}$ is defined by

$$\begin{aligned} F_{\pm} = &\pm (1 - C_{R,L})(w \pm \zeta) + \frac{w^2}{2} \pm \zeta w + \sum_{n=1}^{\infty}(-1)^n \frac{(1+\zeta)^{2n}}{(2n)!} \\ &\times \left[\frac{\partial^{2n} w}{\partial t \partial x^{2n-1}} \pm \frac{(1+\zeta)}{2n+1}\frac{\partial^{2n} w}{\partial x^{2n}} + \frac{1}{2}\sum_{m=0}^{2n}(-1)^m \binom{2n}{m}\frac{\partial^m w}{\partial x^m} \right. \\ &\left. \times \frac{\partial^{2n-m} w}{\partial x^{2n-m}}\right]. \end{aligned} \tag{3.19}$$

For our future purposes it is convenient to introduce the following change of dependent variables

$$w + \zeta = 2\epsilon\alpha, \qquad w - \zeta = -2\epsilon\beta. \tag{3.20}$$

Then, the equation (3.18) takes the following form

$$\begin{aligned} &2\epsilon(C_R + C_L)\left[l\frac{\partial \alpha}{\partial \eta} + \epsilon k l \left(\frac{\partial \theta}{\partial \eta}\frac{\partial \alpha}{\partial \xi} - \frac{\partial \theta}{\partial \xi}\frac{\partial \alpha}{\partial \eta}\right)\right] \\ &+ \left\{k\frac{\partial}{\partial \xi} + l\frac{\partial}{\partial \eta} + \epsilon k l \left[\frac{\partial}{\partial \eta}(\theta - \phi)\frac{\partial}{\partial \xi} - \frac{\partial}{\partial \xi}(\theta - \phi)\frac{\partial}{\partial \eta}\right]\right\} F_{+} = 0. \end{aligned} \tag{3.21}$$

A similar expression is valid for β provided that (α, β), (ξ, η), (k, l), (θ, ϕ) and (F_+, F_-) are replaced with each other. We shall assume that the field quantities may be expanded into asymptotic series in ϵ as follows:

$$\begin{aligned} \alpha(\xi, \eta) &= \alpha_0 + \epsilon\alpha_1 + \epsilon^2\alpha_2 + \dots , \\ \beta(\xi, \eta) &= \beta_0 + \epsilon\beta_1 + \epsilon^2\beta_2 + \dots , \\ \theta(\xi, \eta) &= \epsilon\theta_1 + \epsilon^2\theta_2 + \dots , \\ \phi(\xi, \eta) &= \epsilon\phi_1 + \epsilon^2\phi_2 + \dots , \\ C_R &= 1 + \epsilon a R_1 + \epsilon^2 a^2 R_2 + \dots , \\ C_L &= 1 + \epsilon b L_1 + \epsilon^2 b^2 L_2 + \dots . \end{aligned} \tag{3.22}$$

Here it is to be noted that the terms θ_0 and ϕ_0 in Su and Mirie's [6] work are set equal to zero. This means that in the present work the order of trajectory functions is assumed to be of order ϵ^2.

3.4 Solution of the field equations

Introducing (3.22) into (3.21) and setting the coefficients of like powers of ϵ equal to zero the following sets of differential equations are obtained

$O(\epsilon)$ equations:

$$\frac{\partial \alpha_0}{\partial \eta} = 0, \quad \frac{\partial \beta_0}{\partial \xi} = 0, \tag{3.23}$$

the solution of which yields

$$\alpha_0 = af(\xi), \quad \beta_0 = bg(\eta), \tag{3.24}$$

where $f(\xi)$ and $g(\eta)$ are two unknown functions to be determined from the solution, and a and b are some constants characterizing the wave amplitudes.

$O(\epsilon^2)$ equations:

$$\begin{aligned} &4l\frac{\partial \alpha_1}{\partial \eta} + \frac{1}{3}k^3\alpha_0^{'''} + \frac{2}{3}l^3\beta_0^{'''} - l(\alpha_0 + \beta_0)\beta_0^{'} + (3k\alpha_0 - k\beta_0 \\ &- 2akR_1)\alpha_0^{'} = 0. \end{aligned} \tag{3.25}$$

Integrating equation (3.25) with respect to η and setting the secular terms equal to zero we obtain

$$R_1 = \frac{1}{2}, \quad k^2 = 3a, \quad f^{'''} + 3ff^{'} - f^{'} = 0, \tag{3.26}$$

and

$$\alpha_1 = \frac{7}{8}b^2g^2 + \frac{ab}{4}fg - \frac{b^2}{2}g + a^2F_1(\xi) + \frac{abk}{4l}f^{'}M(\eta), \tag{3.27}$$

where $M(\eta)$ is defined by

$$M(\eta) = \int^{\eta} g(\eta^{'})d\eta^{'}. \tag{3.28}$$

Similar expressions are valid for β_1 by making proper changes between $\alpha_1 \leftrightarrow \beta_1$, $f \leftrightarrow g$, etc.. The result will be as follows

$$L_1 = \frac{1}{2}, \quad l^2 = 3b, \quad g^{'''} + 3gg^{'} - g^{'} = 0, \tag{3.29}$$

and

$$\beta_1 = \frac{7}{8}a^2f^2 + \frac{ab}{4}fg - \frac{a^2}{2}f + b^2G_1(\eta) + \frac{abl}{4k}g^{'}N(\xi), \tag{3.30}$$

where $N(\xi)$ is defined by

$$N(\xi) = \int^{\xi} f(\xi^{'})d\xi^{'}. \tag{3.31}$$

Here $F_1(\xi)$ and $G_1(\eta)$ are two unknown functions whose governing equations are to be obtained from the higher order expansions, R_1 and L_1 are the speed correction terms of order ϵ for the right and left going waves, respectively.

Su and Mirie [6] stated that the terms $f'(\xi)M(\eta)$ in equation (3.27) and $g'(\eta)N(\xi)$ in equation (3.30) do not cause any secularity at this order but they will cause secularity in the next order equations. Therefore, these terms should be eliminated by introducing the functions $\epsilon\theta_0(\eta)$ and $\epsilon\phi_0(\xi)$ in trajectory functions (For further details, see Appendix A). But as will be shown in the solution of the next order differential equations these terms do not cause any secularity; therefore, $\epsilon\theta_0(\eta)$ and $\epsilon\phi_0(\xi)$ must vanish.

$O\left(\epsilon^3\right)$ *equations:*

From the master equation (3.11), for this order, the following equation is obtained

$$
\begin{aligned}
&4l\frac{\partial \alpha_2}{\partial \eta}+ak\frac{\partial^3}{\partial \xi^3}(\alpha_1-\beta_1)-3bk\frac{\partial^3}{\partial \xi\partial \eta^2}(\alpha_1-\beta_1)+k\alpha_0\frac{\partial}{\partial \xi}(3\alpha_1-\beta_1)\\
&-2bl\frac{\partial^3}{\partial \eta^3}(\alpha_1-\beta_1)-l\frac{\partial}{\partial \eta}\left[\beta_0(\alpha_1+\beta_1)\right]-k\beta_0\frac{\partial}{\partial \xi}(\alpha_1+\beta_1)-ak\frac{\partial \alpha_1}{\partial \xi}\\
&+(bl+3l\alpha_0)\frac{\partial \alpha_1}{\partial \eta}+3k\alpha_0^{'}\alpha_1-l\alpha_0\frac{\partial \beta_1}{\partial \eta}-k\alpha_0^{'}\beta_1-\frac{3}{10}a^2k\alpha_0^{(v)}\\
&-\frac{9}{20}b^2l\beta_0^{(v)}+\left(\frac{3}{4}a^2k+3ak\beta_0\right)\alpha_0^{'''}+\left(\frac{3}{4}b^2l+6bl\alpha_0+3bl\beta_0\right)\beta_0^{'''}\\
&+3ak\alpha_0^{'}\alpha_0^{''}+\left(3bk\alpha_0^{'}+6bl\beta_0^{'}\right)\beta_0^{''}+4kl\frac{\partial \theta_1}{\partial \eta}\alpha_0^{'}-2a^2kR_2\alpha_0^{'}=0. \qquad (3.32)
\end{aligned}
$$

When the equation (3.32) is integrated with respect to η there might be two types of secularities. The first type of secularity is of the form $\int^{\eta} M(\eta^{'})d\eta^{'}$ and the second type is proportional to η. Luckily, the coefficient of $\int^{\eta} M(\eta^{'})d\eta^{'}$ in equation (3.32) vanishes identically. This means, as opposed to the statement by Su and Mirie [6], there is no secularity of the type $\int^{\eta} M(\eta^{'})d\eta^{'}$ at this order. Removing the secularity of other type, i.e., the coefficient of η gives the following evolution equation for $F_1(\xi)$

$$
F_1^{'''}+3(fF_1)^{'}-F_1^{'}=(2R_2-\frac{19}{20})f^{'}+\frac{9}{4}ff^{'}+\frac{3}{4}f^2f^{'}. \qquad (3.33)
$$

In order to obtain the localized progressive wave solution for the equation (3.33), one can integrate (3.33) with respect to ξ and use the localization condition, i.e., f and its various order derivatives vanish as $\xi\to\pm\infty$. Then we have

$$
F_1^{''}+(3f-1)F_1=(2R_2-\frac{19}{20})f+\frac{9}{8}f^2+\frac{1}{4}f^3. \qquad (3.34)
$$

As described by Demiray [71, 72], $F_1=f^{'}$ is one of the solutions of the homogeneous equation in (3.34). Therefore, the first term on the right-hand side causes the secularity in the solution of F_1 and the coefficient of f must vanish

$$R_2 = \frac{19}{40}. \tag{3.35}$$

We shall seek a solution for F_1 of the form

$$F_1 = a_1 \operatorname{sech}^4\left(\frac{\xi}{2}\right) + a_2 \operatorname{sech}^2\left(\frac{\xi}{2}\right), \tag{3.36}$$

where a_1 and a_2 are constants to be determined from the solution of (3.34). Taking the derivative of F_1 twice, we obtain

$$F_1^{''} = -5a_1 \operatorname{sech}^6\left(\frac{\xi}{2}\right) + (4a_1 - \frac{3}{2}a_2) \operatorname{sech}^4\left(\frac{\xi}{2}\right) + a_2 \operatorname{sech}^2\left(\frac{\xi}{2}\right). \tag{3.37}$$

Inserting (3.36) and (3.37) into (3.34) and setting the coefficients of $\operatorname{sech}^6\left(\frac{\xi}{2}\right)$ and $\operatorname{sech}^4\left(\frac{\xi}{2}\right)$ equal to zero, one obtains

$$a_1 = -\frac{1}{8}, \quad a_2 = 1. \tag{3.38}$$

Then the particular solution of the differential equation (3.34) reads

$$F_1 = f - \frac{1}{8}f^2. \tag{3.39}$$

Similarly, for the left going wave one obtains

$$L_2 = \frac{19}{40}, \quad G_1 = g - \frac{1}{8}g^2. \tag{3.40}$$

Here R_2 and L_2 are the speed correction terms of order ϵ^2 for the right and left going waves. Introducing (3.39) and (3.40) into the expressions of α_1 and β_1 we have

$$\alpha_1 = \frac{1}{8}(7b^2g^2 - a^2f^2) - \frac{1}{2}(b^2g - 2a^2f) + \frac{ab}{4}fg + \frac{abk}{4l}f^{'}M, \tag{3.41}$$

$$\beta_1 = \frac{1}{8}(7a^2f^2 - b^2g^2) - \frac{1}{2}(a^2f - 2b^2g) + \frac{ab}{4}fg + \frac{abl}{4k}g^{'}N. \tag{3.42}$$

Inserting (3.41) and (3.42) into the equation (3.32) the function α_2 is found to be

$$\begin{aligned}\alpha_2 =& \frac{3}{16}a^2bfg - \frac{9}{8}ab^2fg + 2ab^2fg^2 + \frac{1}{32}a^2bf^2g - \frac{7}{10}b^3g + \frac{43}{32}b^3g^2 \\ &+ \frac{1}{32}b^3g^3 + \frac{3ab^2k^2}{16l^2}fg - \frac{9ab^2k^2}{32l^2}f^2g + \frac{abk}{16l}\left[35af - 9a - b\right] \\ &\times f^{'}M + \frac{abl}{16k}\left[afg^{'} - 2bg^{'} + 7bgg^{'} + \frac{ak}{l}f^{'}g\right]N + \frac{7ab^2k}{16l}f^{'} \\ &\times \int g^2 d\eta^{'} + \frac{ab^2k^2}{16l^2}\left[f - \frac{3}{2}f^2\right]\int gMd\eta^{'} + \frac{ab^2k}{16l}f^{'}\int g^{'}Md\eta^{'} \\ &- akf^{'}\theta_1 + a^3F_2(\xi). \end{aligned} \tag{3.43}$$

A similar expression may be given for β_2. Recalling the expression of $g(\eta)$, i.e., $g = \text{sech}^2\left(\frac{\eta}{2}\right)$ and $M = \int^{\eta} g(\eta^{'})d\eta^{'}$, the following relations may be obtained

$$\begin{aligned} &\int gMd\eta^{'} = -2g, \qquad \int g^{'}Md\eta^{'} = \frac{2}{3}M(g-1), \\ &\int g^2 d\eta^{'} = \frac{2}{3}M(g+2). \end{aligned} \tag{3.44}$$

The integration constants were neglected in the above relations. Since, as can be seen from the equation (3.43), the terms including the integrals given in (3.44) are all functions of the variable ξ except the integrals themselves. If the relations (3.44) are substituted into (3.43) without neglecting the integration constants, then the products containing integration constants will be the functions of ξ only, that is, they can be inserted into the function $F_2(\xi)$. Substituting (3.44) into (3.43) and using the relations $k^2 = 3a$ and $l^2 = 3b$ we have

$$\begin{aligned}\alpha_2 =& \frac{1}{4}a^2bfg - \frac{9}{8}ab^2fg + 2ab^2fg^2 - \frac{1}{16}a^2bf^2g - \frac{7}{10}b^3g \\ &+ \frac{43}{32}b^3g^2 + \frac{1}{32}b^3g^3 + \frac{abk}{16l}\left[35af - 9a + 3b + 3bg\right]f^{'}M \\ &- akf^{'}\theta_1 + \frac{abl}{16k}\left[afg^{'} - 2bg^{'} + 7bgg^{'} + \frac{ak}{l}f^{'}g\right]N + a^3F_2(\xi). \end{aligned} \tag{3.45}$$

By making a proper substitution a similar expression may be given for β_2 as

$$\begin{aligned}\beta_2 =& \frac{1}{4}ab^2fg - \frac{9}{8}a^2bfg + 2a^2bf^2g - \frac{1}{16}ab^2fg^2 - \frac{7}{10}a^3f \\ &+ \frac{43}{32}a^3f^2 + \frac{1}{32}a^3f^3 + \frac{abl}{16k}\left[35bg - 9b + 3a + 3af\right]g^{'}N \\ &- blg^{'}\phi_1 + \frac{abk}{16l}\left[bf^{'}g - 2af^{'} + 7aff^{'} + \frac{bl}{k}fg^{'}\right]M + b^3G_2(\xi). \qquad (3.46)\end{aligned}$$

As can be seen from equations (3.45) and (3.46) the terms $f'M(\eta)$ and $g'N(\xi)$ appearing in the expressions of α_1 and β_1, respectively, do not cause any secularity in the solution of α_2 and β_2. Therefore the statement by Su and Mirie [6] is incorrect. However, as was stated before, some of the terms appearing in the expressions of α_2 and β_2 (The equations (3.45) and (3.46)) may cause additional secularity in the expressions of α_3 and β_3.

There appears to be two types of secularity in the solution of $\mathcal{O}(\epsilon^4)$ equation. As was seen before, the first type of secularity results from the terms proportional to ξ and η which will be studied later. The second type of secularity occurs from the terms proportional $\int^{\xi} N(\xi^{'})d\xi^{'}$ and $\int^{\eta} M(\eta^{'})d\eta^{'}$ as $\xi(\eta) \to \pm\infty$. Here we shall first consider only the parts of $\mathcal{O}(\epsilon^4)$ equations leading to $\int^{\eta} M(\eta^{'},\tau)d\eta^{'}$ type of secularity. Similar expressions may be valid for $\int^{\xi} N(\xi^{'},\tau)d\xi^{'}$ type of secularity.

For this purpose we consider the following part of the $\mathcal{O}(\epsilon^4)$ equation (The complete $\mathcal{O}(\epsilon^4)$ equation is given in Appendix B).

$$\begin{aligned}&4l\frac{\partial\alpha_3}{\partial\eta} + ak\frac{\partial^3}{\partial\xi^3}(\alpha_2 - \beta_2) - ak\frac{\partial\alpha_2}{\partial\xi} + k\frac{\partial}{\partial\xi}(\alpha_0[3\alpha_2 - \beta_2]) \\ &+ 3ak\alpha_0^{'}\frac{\partial^2}{\partial\xi^2}(\alpha_1 - \beta_1) - \frac{3}{10}a^2k\frac{\partial^5}{\partial\xi^5}(\alpha_1 - \beta_1) + \frac{3}{4}a^2k\frac{\partial^3}{\partial\xi^3}(\alpha_1 - \beta_1) \\ &+ 3ak\alpha_0^{''}\frac{\partial\alpha_1}{\partial\xi} - \frac{19}{20}a^2k\frac{\partial\alpha_1}{\partial\xi} + k\alpha_1\frac{\partial}{\partial\xi}(3\alpha_1 - \beta_1) - k\beta_1\frac{\partial}{\partial\xi}(\alpha_1 + \beta_1) \\ &+ 3ak\alpha_0^{'''}\beta_1 + \frac{9}{280}a^3k\alpha_0^{(vii)} - \frac{3}{16}a^3k\alpha_0^{(v)} - \frac{3}{4}a^2k\alpha_0\alpha_0^{(v)}\end{aligned}$$

$$
-\frac{9}{4}a^2k\alpha_0^{'}\alpha_0^{(iv)} - 3ak\alpha_0^2\alpha_0^{'''} + \frac{3}{4}a^2k\alpha_0^{''}\alpha_0^{'''} + \frac{57}{80}a^3k\alpha_0^{'''} + 3ak\left(\alpha_0^{'}\right)^3
+\frac{3}{2}a^2k\frac{\partial}{\partial\xi}(\alpha_0\alpha_0^{''}) - 2R_3a^3k\alpha_0^{'} = 0. \tag{3.47}
$$

A similar expression can be given for $4l\frac{\partial\beta_3}{\partial\xi}$. We split (3.47) into two parts which contain the variables α_2 and (α_1, β_1, β_2), respectively. Then, we obtain

$$
ak\frac{\partial^3\alpha_2}{\partial\xi^3} - ak\frac{\partial\alpha_2}{\partial\xi} + 3k\frac{\partial}{\partial\xi}(\alpha_0\alpha_2) = \frac{35a^3bk^2}{16l}\left(\frac{63}{4}f^4 - 21f^3 + 6f^2\right), \tag{3.48}
$$

$$
-\frac{3}{10}a^2k\frac{\partial^5\alpha_1}{\partial\xi^5} + 3ak\frac{\partial}{\partial\xi}\left(\alpha_0^{'}\frac{\partial\alpha_1}{\partial\xi}\right) + \frac{3}{4}a^2k\frac{\partial^3\alpha_1}{\partial\xi^3} - \frac{19}{20}a^2k\frac{\partial\alpha_1}{\partial\xi}
+3k\alpha_1\frac{\partial\alpha_1}{\partial\xi} - k\frac{\partial}{\partial\xi}(\alpha_1\beta_1) - ak\frac{\partial^3\beta_2}{\partial\xi^3} - k\frac{\partial}{\partial\xi}(\alpha_0\beta_2)
= \frac{a^3bk^2}{16l}\left(\frac{63}{4}f^4 - 21f^3 + 6f^2\right), \tag{3.49}
$$

where the identity $(f^{'})^2 = f^2 - f^3$ is used. As might be seen from the equations (3.48) and (3.49), the integration of these equations with respect to η cause secularity in the expression of α_3. Similar type of secularity also occurs in the expression of β_3. In order to remove the secularities, the trajectory functions θ_1 and ϕ_1 should be in the following form

$$
\theta_1 = \frac{9ab}{4l}f\int_{-\infty}^{\eta} g(\eta^{'})d\eta^{'}, \quad \phi_1 = \frac{9ab}{4k}g\int_{+\infty}^{\xi} f(\xi^{'})d\xi^{'}. \tag{3.50}
$$

To remove the secularities of the type η, one can use the equation (3.47) to obtain the evolution equation for $F_2(\xi)$. In order to remove the secularity, the following equation must be satisfied

$$
F_2^{'''} + (3fF_2)^{'} - (F_2)^{'} = S^{'}(f), \tag{3.51}
$$

where $S(f)$ is defined as follows

$$
S(f) = (2R_3 - \frac{55}{56})f - \frac{591}{64}f^4 + \left(\frac{201}{16} + \frac{3}{8a}\right)f^3 - \left(\frac{393}{160} + \frac{3}{8a}\right)f^2. \tag{3.52}
$$

Integrating (3.51) with respect to ξ and using localization condition, we obtain

$$F_2^{''} + (3f-1)F_2 = (2R_3 - \frac{55}{56})f - \frac{591}{64}f^4 + \left(\frac{201}{16} + \frac{3}{8a}\right)f^3 - \left(\frac{393}{160} + \frac{3}{8a}\right)f^2. \tag{3.53}$$

Since the first term in the right-hand side of (3.53) causes the secularity in the solution of F_2, the coefficient of f must vanish

$$R_3 = \frac{55}{112}. \tag{3.54}$$

We shall propose a solution for F_2 of the following form

$$F_2 = b_1 \operatorname{sech}^6\left(\frac{\xi}{2}\right) + b_2 \operatorname{sech}^4\left(\frac{\xi}{2}\right) + b_3 \operatorname{sech}^2\left(\frac{\xi}{2}\right) \tag{3.55}$$

where b_i are constants to be determined from the solution of (3.53). Carrying out the derivative of F_2 we obtain

$$F_2 = -\frac{21}{2}b_1 \operatorname{sech}^8\left(\frac{\xi}{2}\right) + (9b_1 - 5b_2)\operatorname{sech}^6\left(\frac{\xi}{2}\right) + (4b_2 - \frac{3}{2}b_3)\operatorname{sech}^4\left(\frac{\xi}{2}\right) + b_3 \operatorname{sech}^2\left(\frac{\xi}{2}\right). \tag{3.56}$$

Inserting (3.55) and (3.56) into (3.53) and setting the coefficients of $\operatorname{sech}^8\left(\frac{\xi}{2}\right)$, $\operatorname{sech}^6\left(\frac{\xi}{2}\right)$ and $\operatorname{sech}^4\left(\frac{\xi}{2}\right)$ equal to zero, we have

$$b_1 = \frac{197}{160}, \quad b_2 = -\left(\frac{217}{160} + \frac{3}{16a}\right), \quad b_3 = \frac{43}{40} + \frac{1}{8a}. \tag{3.57}$$

Then the particular solution of the differential equation (3.53) can be written as

$$F_2 = \frac{197}{160}f^3 - \left(\frac{217}{160} + \frac{3}{16a}\right)f^2 + \left(\frac{43}{40} + \frac{1}{8a}\right)f. \tag{3.58}$$

Similarly, for other unknowns we have

$$L_3 = \frac{55}{112}, \qquad G_2 = \frac{197}{160}g^3 - \left(\frac{217}{160} + \frac{3}{16b}\right)g^2 + \left(\frac{43}{40} + \frac{1}{8b}\right)g. \tag{3.59}$$

Here R_3 and L_3 correspond to ϵ^3 order speed correction terms of the right and left going waves. Then, the final solution for α_2 and β_2 take the following form

$$\begin{aligned}\alpha_2 =& \frac{1}{4}a^2bfg - \frac{9}{8}ab^2fg + 2ab^2fg^2 - \frac{1}{16}a^2bf^2g + \frac{1}{32}b^3g^3 + \frac{43}{32}b^3g^2 \\ & - \frac{7}{10}b^3g + \frac{197}{160}a^3f^3 - \left(\frac{217}{160}a^3 + \frac{3}{16}a^2\right)f^2 + \left(\frac{43}{40}a^3 + \frac{1}{8}a^2\right) \\ & \times f + \frac{abk}{16l}\left(-af - 9a + 3b + 3bg\right)f^{'}M + \frac{abl}{16k}\left(afg^{'} - 2bg^{'}\right. \\ & \left. + 7bgg^{'} + \frac{ak}{l}f^{'}g\right)N, \end{aligned} \tag{3.60}$$

$$\begin{aligned}\beta_2 =& \frac{1}{4}ab^2fg - \frac{9}{8}a^2bfg + 2a^2bf^2g - \frac{1}{16}ab^2fg^2 + \frac{1}{32}a^3f^3 + \frac{43}{32}a^3f^2 \\ & - \frac{7}{10}a^3f + \frac{197}{160}b^3g^3 - \left(\frac{217}{160}b^3 + \frac{3}{16}b^2\right)g^2 + \left(\frac{43}{40}b^3 + \frac{1}{8}b^2\right) \\ & \times g + \frac{abl}{16k}\left(-bg - 9b + 3a + 3af\right)g^{'}N + \frac{abk}{16l}\left(bf^{'}g - 2af^{'}\right. \\ & \left. + 7aff^{'} + \frac{bl}{k}fg^{'}\right)M. \end{aligned} \tag{3.61}$$

Thus, for this order, the trajectories of the solitary waves become

$$\begin{aligned} \epsilon^{\frac{1}{2}}k(x - C_Rt) &= \xi - \epsilon^2 k\theta_1 + \mathcal{O}(\epsilon^3), \\ \epsilon^{\frac{1}{2}}l(x + C_Lt) &= \eta - \epsilon^2 l\phi_1 + \mathcal{O}(\epsilon^3). \end{aligned} \tag{3.62}$$

3.4.1 Phase Shifts

To obtain the phase shifts after the head-on collision of solitary waves characterized by a and b are asymptotically far from each other at the initial time ($t = -\infty$), the solitary wave a is at $\xi = 0$, $\eta = -\infty$, and the solitary wave b is at $\eta = 0$, $\xi = +\infty$, respectively. After the collision ($t = +\infty$), the solitary wave b is far to the right of solitary wave a, i.e., the solitary wave a is at $\xi = 0$, $\eta = +\infty$, and the solitary wave b is at $\eta = 0$, $\xi = -\infty$. Using (3.50) and (3.62) one can obtain the corresponding phase shifts Δ_a and Δ_b as follows:

$$\begin{aligned}
\Delta_a &= \epsilon^{1/2} k(x - C_R t)\mid_{\xi=0,\eta=\infty} - \epsilon^{1/2} k(x - C_R t)\mid_{\xi=0,\eta=-\infty} \\
&= -\epsilon^2 \frac{kab}{4l} 9 f(0) \int_{-\infty}^{+\infty} g(\eta') d\eta' \\
&= -\epsilon^2 \frac{9kab}{4l} \int_{-\infty}^{+\infty} g(\eta') d\eta', \qquad (3.63) \\
\Delta_b &= \epsilon^{1/2} k(x + C_L t)\mid_{\eta=0,\xi=-\infty} - \epsilon^{1/2} k(x + C_L t)\mid_{\eta=0,\xi=\infty} \\
&= \epsilon^2 \frac{lab}{4k} 9 g(0) \int_{-\infty}^{+\infty} f(\xi') d\xi' \\
&= \epsilon^2 \frac{9lab}{4k} \int_{-\infty}^{+\infty} f(\xi') d\xi'. \qquad (3.64)
\end{aligned}$$

Using the explicit expressions of $f(\xi)$ and $g(\eta)$ the phase shifts are obtained as

$$\Delta_a = -\epsilon^2 \frac{9kab}{l}, \quad \Delta_b = \epsilon^2 \frac{9lab}{k}. \qquad (3.65)$$

Here, as opposed to the results of previous works on the same subject the phase shifts depend on the amplitudes of both waves.

3.5 Result and Conclusion

Starting with non-dimensional field equations (3.11) and (3.12), introducing the stretched coordinates with trajectory functions of order ϵ^2 and expanding the field variables and trajectory functions into power series in ϵ, we obtained a set of differential equations governing the various terms in perturbation expansion. By solving these differential equations under the restriction of non-secular solution we obtained evolution equations governing the colliding solitary waves and trajectory functions. Remembering the change of variables (3.20), the surface elevation parameter ζ and the axial velocity parameter w may be given as follows

$$\zeta = \epsilon \Bigg\{ af + bg + \epsilon \left[\frac{3}{4}\left(a^2f^2+b^2g^2\right) + \frac{1}{2}\left(a^2f+b^2g\right) + \frac{1}{2}abfg \right.$$
$$\left. + \frac{abk}{4l}f^{'}M + \frac{abl}{4k}g^{'}N \right] + \epsilon^2 \left[\frac{101}{80}\left(a^3f^3+b^3g^3\right) - \frac{1}{80}\left(a^3f^2 \right.\right.$$
$$\left. +b^3g^2\right) - \frac{3}{16}\left(a^2f^2+b^2g^2\right) + \frac{3}{8}\left(a^3f+b^3g\right) + \frac{1}{8}\left(a^2f+b^2g\right)$$
$$+\frac{31}{16}ab\left(af^2g+bfg^2\right) - \frac{7}{8}ab(a+b)fg + \frac{abk}{16l}\Bigg(6aff^{'} - (11a$$
$$-3b)f^{'} + 4bf^{'}g + \frac{bl}{k}fg^{'}\Bigg)M + \frac{abl}{16k}\Bigg(6bgg^{'} - (11b-3a)g^{'}$$
$$\left. +4afg^{'} + \frac{ak}{l}f^{'}g\Bigg)N \right] + \dots \Bigg\}, \tag{3.66}$$

$$w = \epsilon \Bigg\{ af - bg + \epsilon \left[\left(-a^2f^2+b^2g^2\right) + \frac{3}{2}\left(a^2f-b^2g\right) + \frac{abk}{4l}f^{'}M \right.$$
$$\left. -\frac{abl}{4k}g^{'}N \right] + \epsilon^2 \left[\frac{6}{5}\left(a^3f^3-b^3g^3\right) - \frac{27}{10}\left(a^3f^2-b^3g^2\right) \right.$$
$$-\frac{3}{16}\left(a^2f^2-b^2g^2\right) + \frac{71}{40}\left(a^3f-b^3g\right) + \frac{1}{8}\left(a^2f-b^2g\right)$$
$$-\frac{33}{16}ab\left(af^2g-bfg^2\right) + \frac{11}{8}ab(a-b)fg + \frac{abk}{16l}\Bigg(- 8aff^{'}$$
$$-(7a-3b)f^{'} + 2bf^{'}g - \frac{bl}{k}fg^{'}\Bigg)M + \frac{abl}{16k}\Bigg(8bgg^{'} + (7b-3a)g^{'}$$
$$\left. -2afg^{'} + \frac{ak}{l}f^{'}g\Bigg)N \right] + \dots \Bigg\}. \tag{3.67}$$

Using the conventional definition of phase shifts we obtained the explicit expressions of them. As opposed to the result of previous works on the same subject in our case the phase shifts are found to be depend on amplitudes of both waves. We further noticed that the order of phase shift is ϵ^2 rather than ϵ.

Chapter 4

Head-on Collision Between Two Solitary Waves in Shallow Water: The Use of the Extended PLK Method

4.1 Introduction

In Chapter 3, we have studied head-on collision problem between two solitary waves in shallow water through the use of strained coordinates of the form

$$\epsilon^{\frac{1}{2}}k(x - C_R t) = \xi - \epsilon k\theta(\xi, \eta),$$
$$\epsilon^{\frac{1}{2}}l(x + C_L t) = \eta - \epsilon l\phi(\xi, \eta),$$

and ended up with a set of ordinary differential equations as the evolution equation whose solutions gives approximate progressive wave solutions to the field equations. In practice, we are concerned with approximate solutions, not necessarily of progressive type. In this case we need evolution equations contain both space and time derivatives. For that purpose we shall introduce a different set of stretched coordinates so called the extended PLK method which is the combination of the classical reductive perturbation method and the strained coordinates. We introduce the stretched coordinates as

$$\begin{aligned}
\epsilon^{1/2}(x - t) =& \xi + \epsilon p(\tau) + \epsilon^2 P(\xi, \eta, \tau),\\
\epsilon^{1/2}(x + t) =& \eta + \epsilon q(\tau) + \epsilon^2 Q(\xi, \eta, \tau),\\
\epsilon^{3/2}t =& \tau,
\end{aligned}$$

where ϵ is the smallness parameter measuring the weakness of dispersion and nonlinearity, $p(\tau)$ and $q(\tau)$ are two unknown functions characterizing the higher order dispersive effects (Demiray [71, 72]), $P(\xi, \eta, \tau)$ and $Q(\xi, \eta, \tau)$ are two unknown functions characterizing the phase shifts after collision. These unknown functions are to be determined from the higher order perturbation expansions so as to remove possible secularities that might occur in the solution.

Expanding the field variables and these unknown functions into power series of ϵ, introducing these expansion into the field equations and setting the coefficients of various powers of ϵ equal to zero we obtained a set of partial differential equations. By solving these differential equations and removing possible secularities that might occur in the solution we obtained various order evolution equations and restrictions that make it possible to determine the unknown functions. Seeking a progressive wave solution to these evolution equations we obtained the speed correction terms and the phase shifts. It is observed that the result found here is exactly the same with one obtained in the previous chapter. The variations of the wave profiles of right-going waves before and after the collision are depicted on Figure 4.1. It is seen that the wave profile before the collision is symmetric, whereas after the collision it is unsymmetrical and tilts backward with respect to the direction of its propagation.

4.2 Basic Equations

To study the head-on collision problem in shallow water theory, the equations (3.11) and (3.12) can be rewritten as follows:

$$\frac{\partial \hat{\zeta}}{\partial t} + \frac{\partial}{\partial x}\left\{(1+\hat{\zeta})\hat{w} + \sum_{n=1}^{\infty}(-1)^n \frac{(1+\hat{\zeta})^{2n+1}}{(2n+1)!}\frac{\partial^{2n}\hat{w}}{\partial x^{2n}}\right\} = 0, \tag{4.1}$$

$$\frac{\partial \hat{w}}{\partial t} + \frac{\partial}{\partial x}\left\{\hat{\zeta} + \frac{\hat{w}^2}{2} + \sum_{n=1}^{\infty}(-1)^n \frac{(1+\hat{\zeta})^{2n}}{(2n)!}\left[\frac{\partial^{2n}\hat{w}}{\partial t \partial x^{2n-1}} + \frac{1}{2}\sum_{m=0}^{2n}(-1)^m \binom{2n}{m}\frac{\partial^m \hat{w}}{\partial x^m}\frac{\partial^{2n-m}\hat{w}}{\partial x^{2n-m}}\right]\right\} = 0, \tag{4.2}$$

where $\hat{w} = \dfrac{\partial \hat{\Phi}}{\partial x}$ and $\dbinom{2n}{m}$ is the binomial coefficient. Here, the variables defined in Chapter 3 are relabelled as $\zeta \to \hat{\zeta}$, $\psi \to \hat{\phi}$ and $\Psi \to \hat{\Phi}$.

4.3 Extended PLK Method

For our future purposes, we introduce the following stretched coordinates

$$\begin{aligned}
\epsilon^{\frac{1}{2}}(x-t) =& \xi + \epsilon p(\tau) + \epsilon^2 P(\xi,\eta,\tau),\\
\epsilon^{\frac{1}{2}}(x+t) =& \eta + \epsilon q(\tau) + \epsilon^2 Q(\xi,\eta,\tau),\\
\epsilon^{3/2} t =& \tau,
\end{aligned} \tag{4.3}$$

where ϵ is the smallness parameter measuring the weakness of dispersion and nonlinearity, $p(\tau)$ and $q(\tau)$ are two unknown functions characterizing the higher order dispersive effects, $P(\xi,\eta,\tau)$ and $Q(\xi,\eta,\tau)$ are two unknown functions characterizing the phase shifts after collision. Then, the following differential relations hold true

$$\begin{aligned}
\frac{\partial}{\partial x} =& \frac{\epsilon^{\frac{1}{2}}}{D}\left\{\left[1+\epsilon^2\left(\frac{\partial Q}{\partial \eta}-\frac{\partial P}{\partial \eta}\right)\right]\frac{\partial}{\partial \xi}+\left[1+\epsilon^2\left(\frac{\partial P}{\partial \xi}-\frac{\partial Q}{\partial \xi}\right)\right]\frac{\partial}{\partial \eta}\right\},\\
\frac{\partial}{\partial t} =& \epsilon^{1/2}\left\{\epsilon\frac{\partial}{\partial \tau}-\frac{1}{D}\left[1+\epsilon^2\left(\frac{dp}{d\tau}+\frac{\partial P}{\partial \eta}+\frac{\partial Q}{\partial \eta}\right)+\epsilon^3\frac{\partial P}{\partial \tau}+\right.\right.\\
&\left.\epsilon^4\left(\frac{dp}{d\tau}\frac{\partial Q}{\partial \eta}-\frac{dq}{d\tau}\frac{\partial P}{\partial \eta}\right)+\epsilon^5\left(\frac{\partial P}{\partial \tau}\frac{\partial Q}{\partial \eta}-\frac{\partial Q}{\partial \tau}\frac{\partial P}{\partial \eta}\right)\right]\frac{\partial}{\partial \xi}\\
&+\frac{1}{D}\left[1+\epsilon^2\left(-\frac{dq}{d\tau}+\frac{\partial P}{\partial \xi}+\frac{\partial Q}{\partial \xi}\right)-\epsilon^3\frac{\partial Q}{\partial \tau}+\epsilon^4\left(\frac{dp}{d\tau}\frac{\partial Q}{\partial \xi}\right.\right.\\
&\left.\left.\left.-\frac{dq}{d\tau}\frac{\partial P}{\partial \xi}\right)+\epsilon^5\left(\frac{\partial P}{\partial \tau}\frac{\partial Q}{\partial \xi}-\frac{\partial Q}{\partial \tau}\frac{\partial P}{\partial \xi}\right)\right]\frac{\partial}{\partial \eta}\right\}
\end{aligned} \tag{4.4}$$

where D is defined by

$$D = \left(1+\epsilon^2\frac{\partial P}{\partial \xi}\right)\left(1+\epsilon^2\frac{\partial Q}{\partial \eta}\right)-\epsilon^4\frac{\partial P}{\partial \eta}\frac{\partial Q}{\partial \xi}. \tag{4.5}$$

We assume that the field quantities $\hat{w}$, $\hat{\zeta}$, $p(\tau)$, $q(\tau)$, $P(\xi,\eta,\tau)$ and $Q(\xi,\eta,\tau)$

can be expanded into asymptotic series in ϵ as

$$\begin{aligned}
&\hat{w} = \epsilon\left[w_0 + \epsilon w_1 + \epsilon^2 w_2 + \epsilon^3 w_3 + \epsilon^4 w_4 + ...\right], \\
&\hat{\zeta} = \epsilon\left[\zeta_0 + \epsilon\zeta_1 + \epsilon^2\zeta_2 + \epsilon^3\zeta_3 + \epsilon^4\zeta_4 + ...\right], \\
&p(\tau) = p_0(\tau) + \epsilon p_1(\tau) + \epsilon^2 p_2(\tau) + \epsilon^3 p_3(\tau) + ... , \\
&q(\tau) = q_0(\tau) + \epsilon q_1(\tau) + \epsilon^2 q_2(\tau) + \epsilon^3 q_3(\tau) + ... , \\
&P(\xi,\eta,\tau) = P_0(\xi,\eta,\tau) + \epsilon P_1(\xi,\eta,\tau) + ... , \\
&Q(\xi,\eta,\tau) = Q_0(\xi,\eta,\tau) + \epsilon Q_1(\xi,\eta,\tau) + ...
\end{aligned} \tag{4.6}$$

Inserting (4.4) and (4.6) into equations (4.1) and (4.2) and setting the coefficients of like powers of ϵ equal to zero the following equations are obtained

$\mathcal{O}(\epsilon)$ *equations:*

$$\begin{aligned}
&\frac{\partial\zeta_0}{\partial\eta} - \frac{\partial\zeta_0}{\partial\xi} + \frac{\partial w_0}{\partial\eta} + \frac{\partial w_0}{\partial\xi} = 0, \\
&\frac{\partial\zeta_0}{\partial\eta} + \frac{\partial\zeta_0}{\partial\xi} + \frac{\partial w_0}{\partial\eta} - \frac{\partial w_0}{\partial\xi} = 0,
\end{aligned} \tag{4.7}$$

$\mathcal{O}\left(\epsilon^2\right)$ *equations:*

$$\begin{aligned}
&\frac{\partial\zeta_1}{\partial\eta} - \frac{\partial\zeta_1}{\partial\xi} + \frac{\partial w_1}{\partial\eta} + \frac{\partial w_1}{\partial\xi} + \frac{\partial\zeta_0}{\partial\tau} + \frac{\partial}{\partial\eta}(\zeta_0 w_0) + \frac{\partial}{\partial\xi}(\zeta_0 w_0) \\
&- \frac{1}{6}\left(\frac{\partial^3 w_0}{\partial\xi^3} + 3\frac{\partial^3 w_0}{\partial\xi^2\partial\eta} + 3\frac{\partial^3 w_0}{\partial\xi\partial\eta^2} + \frac{\partial^3 w_0}{\partial\eta^3}\right) = 0, \\
&\frac{\partial\zeta_1}{\partial\eta} + \frac{\partial\zeta_1}{\partial\xi} + \frac{\partial w_1}{\partial\eta} - \frac{\partial w_1}{\partial\xi} + \frac{\partial w_0}{\partial\tau} + \frac{1}{2}\frac{\partial}{\partial\eta}(w_0^2) + \frac{1}{2}\frac{\partial}{\partial\xi}(w_0^2) \\
&+ \frac{1}{2}\left(\frac{\partial^3 w_0}{\partial\xi^3} + \frac{\partial^3 w_0}{\partial\xi^2\partial\eta} - \frac{\partial^3 w_0}{\partial\xi\partial\eta^2} - \frac{\partial^3 w_0}{\partial\eta^3}\right) = 0,
\end{aligned} \tag{4.8}$$

$\mathcal{O}\left(\epsilon^3\right)$ *equations:*

$$\frac{\partial \zeta_2}{\partial \eta} - \frac{\partial \zeta_2}{\partial \xi} + \frac{\partial w_2}{\partial \eta} + \frac{\partial w_2}{\partial \xi} + \frac{\partial \zeta_1}{\partial \tau} + \frac{\partial}{\partial \eta}(\zeta_1 w_0) + \frac{\partial}{\partial \xi}(\zeta_1 w_0) + \frac{\partial}{\partial \eta}(\zeta_0 w_1)$$
$$+ \frac{\partial}{\partial \xi}(\zeta_0 w_1) - \frac{1}{6}\left(\frac{\partial^3 w_1}{\partial \xi^3} + 3\frac{\partial^3 w_1}{\partial \xi^2 \partial \eta} + 3\frac{\partial^3 w_1}{\partial \xi \partial \eta^2} + \frac{\partial^3 w_1}{\partial \eta^3}\right) - \frac{dq_0}{d\tau}\frac{\partial \zeta_0}{\partial \eta}$$
$$- \frac{dp_0}{d\tau}\frac{\partial \zeta_0}{\partial \xi} + \frac{1}{120}\left(\frac{\partial^5 w_0}{\partial \xi^5} + 5\frac{\partial^5 w_0}{\partial \xi^4 \partial \eta} + 10\frac{\partial^5 w_0}{\partial \xi^3 \partial \eta^2} + 10\frac{\partial^5 w_0}{\partial \xi^2 \partial \eta^3}\right.$$
$$\left. +5\frac{\partial^5 w_0}{\partial \xi \partial \eta^4} + \frac{\partial^5 w_0}{\partial \eta^5}\right) - \frac{\zeta_0}{2}\left(\frac{\partial^3 w_0}{\partial \xi^3} + 3\frac{\partial^3 w_0}{\partial \xi^2 \partial \eta} + 3\frac{\partial^3 w_0}{\partial \xi \partial \eta^2} + \frac{\partial^3 w_0}{\partial \eta^3}\right)$$
$$- \left(6\frac{\partial P_0}{\partial \xi} + 7\frac{\partial Q_0}{\partial \eta} - 6\frac{\partial P_0}{\partial \eta}\frac{\partial Q_0}{\partial \xi}\right)\frac{\partial}{\partial \xi}(\zeta_0 - w_0) + \frac{\partial Q_0}{\partial \xi}\frac{\partial}{\partial \eta}(\zeta_0 - w_0)$$
$$- \frac{\partial P_0}{\partial \eta}\frac{\partial}{\partial \xi}(\zeta_0 + w_0) + \left(7\frac{\partial P_0}{\partial \xi} + 6\frac{\partial Q_0}{\partial \eta} - 6\frac{\partial P_0}{\partial \eta}\frac{\partial Q_0}{\partial \xi}\right)\frac{\partial}{\partial \eta}(\zeta_0 + w_0)$$
$$- \frac{1}{2}\left(\frac{\partial^2 w_0}{\partial \xi^2} + 2\frac{\partial^2 w_0}{\partial \xi \partial \eta} + \frac{\partial^2 w_0}{\partial \eta^2}\right)\left(\frac{\partial \zeta_0}{\partial \eta} + \frac{\partial \zeta_0}{\partial \xi}\right) = 0,$$

$$\frac{\partial \zeta_2}{\partial \eta} + \frac{\partial \zeta_2}{\partial \xi} + \frac{\partial w_2}{\partial \eta} - \frac{\partial w_2}{\partial \xi} + \frac{\partial w_1}{\partial \tau} + \frac{\partial}{\partial \eta}(w_0 w_1) + \frac{\partial}{\partial \xi}(w_0 w_1)$$
$$- \frac{dq_0}{d\tau}\frac{\partial w_0}{\partial \eta} - \frac{dp_0}{d\tau}\frac{\partial w_0}{\partial \xi} + \frac{1}{2}\left(\frac{\partial^3 w_1}{\partial \xi^3} + \frac{\partial^3 w_1}{\partial \xi^2 \partial \eta} - \frac{\partial^3 w_1}{\partial \xi \partial \eta^2} - \frac{\partial^3 w_1}{\partial \eta^3}\right)$$
$$- \frac{1}{24}\left(\frac{\partial^5 w_0}{\partial \xi^5} + 3\frac{\partial^5 w_0}{\partial \xi^4 \partial \eta} + 2\frac{\partial^5 w_0}{\partial \xi^3 \partial \eta^2} - 2\frac{\partial^5 w_0}{\partial \xi^2 \partial \eta^3} - 3\frac{\partial^5 w_0}{\partial \xi \partial \eta^4}\right.$$
$$\left. -\frac{\partial^5 w_0}{\partial \eta^5}\right) + \frac{1}{2}\left(\frac{\partial^2 w_0}{\partial \xi^2} + 2\frac{\partial^2 w_0}{\partial \xi \partial \eta} + \frac{\partial^2 w_0}{\partial \eta^2}\right)\left(\frac{\partial w_0}{\partial \eta} + \frac{\partial w_0}{\partial \xi}\right)$$
$$- \frac{w_0}{2}\left(\frac{\partial^3 w_0}{\partial \xi^3} + 3\frac{\partial^3 w_0}{\partial \xi^2 \partial \eta} + 3\frac{\partial^3 w_0}{\partial \xi \partial \eta^2} + \frac{\partial^3 w_0}{\partial \eta^3}\right)$$
$$+ \zeta_0\left(\frac{\partial^3 w_0}{\partial \xi^3} + \frac{\partial^3 w_0}{\partial \xi^2 \partial \eta} - \frac{\partial^3 w_0}{\partial \xi \partial \eta^2} - \frac{\partial^3 w_0}{\partial \eta^3}\right)$$
$$- \frac{1}{2}\frac{\partial}{\partial \tau}\left[\frac{\partial^2 w_0}{\partial \xi^2} + 2\frac{\partial^2 w_0}{\partial \xi \partial \eta} + \frac{\partial^2 w_0}{\partial \eta^2}\right] - \frac{\partial Q_0}{\partial \xi}\frac{\partial}{\partial \eta}(\zeta_0 - w_0)$$
$$- \frac{\partial P_0}{\partial \eta}\frac{\partial}{\partial \xi}(\zeta_0 + w_0) + \left(7\frac{\partial P_0}{\partial \xi} + 6\frac{\partial Q_0}{\partial \eta} - 6\frac{\partial P_0}{\partial \eta}\frac{\partial Q_0}{\partial \xi}\right)\frac{\partial}{\partial \eta}(\zeta_0 + w_0)$$
$$+ \left(6\frac{\partial P_0}{\partial \xi} + 7\frac{\partial Q_0}{\partial \eta} - 6\frac{\partial P_0}{\partial \eta}\frac{\partial Q_0}{\partial \xi}\right)\frac{\partial}{\partial \xi}(\zeta_0 - w_0)$$

$$+\left(\frac{\partial^2 w_0}{\partial \xi^2}-\frac{\partial^2 w_0}{\partial \eta^2}\right)\left(\frac{\partial \zeta_0}{\partial \eta}+\frac{\partial \zeta_0}{\partial \xi}\right)=0. \tag{4.9}$$

To save the space of the main body of the text, $\mathcal{O}\left(\epsilon^4\right)$ equations are given in Appendix C.

4.4 Solution of the field equations

From the solution of the set (4.7) we have

$$\begin{aligned}\zeta_0 &= f(\xi,\tau)+g(\eta,\tau),\\ w_0 &= f(\xi,\tau)-g(\eta,\tau),\end{aligned} \tag{4.10}$$

where $f(\xi,\tau)$ and $g(\eta,\tau)$ are two unknown functions whose governing equations will be obtained later.

The solution of (4.8) yields

$$\begin{aligned}&2\frac{\partial}{\partial \eta}(\zeta_1+w_1)+2\frac{\partial f}{\partial \tau}+3f\frac{\partial f}{\partial \xi}+\frac{1}{3}\frac{\partial^3 f}{\partial \xi^3}-g\frac{\partial g}{\partial \eta}+\frac{2}{3}\frac{\partial^3 g}{\partial \eta^3}\\ &-f\frac{\partial g}{\partial \eta}-\frac{\partial f}{\partial \xi}g=0,\end{aligned} \tag{4.11}$$

$$\begin{aligned}&2\frac{\partial}{\partial \xi}(\zeta_1-w_1)-2\frac{\partial g}{\partial \tau}+3g\frac{\partial g}{\partial \eta}+\frac{1}{3}\frac{\partial^3 g}{\partial \eta^3}-f\frac{\partial f}{\partial \xi}+\frac{2}{3}\frac{\partial^3 f}{\partial \xi^3}\\ &-g\frac{\partial f}{\partial \xi}-\frac{\partial g}{\partial \eta}f=0.\end{aligned} \tag{4.12}$$

Integrating (4.11) with respect to η and (4.12) with respect to ξ we obtain

$$\begin{aligned}&2(\zeta_1+w_1)+\eta\left[2\frac{\partial f}{\partial \tau}+3f\frac{\partial f}{\partial \xi}+\frac{1}{3}\frac{\partial^3 f}{\partial \xi^3}\right]-\frac{g^2}{2}+\frac{2}{3}\frac{\partial^2 g}{\partial \eta^2}\\ &-fg-M(\eta,\tau)\frac{\partial f}{\partial \xi}=4F_1(\xi,\tau),\end{aligned} \tag{4.13}$$

$$\begin{aligned}&2(\zeta_1-w_1)-\xi\left[2\frac{\partial g}{\partial \tau}-3g\frac{\partial g}{\partial \eta}-\frac{1}{3}\frac{\partial^3 g}{\partial \eta^3}\right]-\frac{f^2}{2}+\frac{2}{3}\frac{\partial^2 f}{\partial \xi^2}\\ &-fg-N(\xi,\tau)\frac{\partial g}{\partial \eta}=4G_1(\eta,\tau),\end{aligned} \tag{4.14}$$

where $F_1(\xi,\tau)$ and $G_1(\eta,\tau)$ are new unknown functions, $M(\eta,\tau)$ and $N(\xi,\tau)$ are defined by

$$M(\eta,\tau)=\int^{\eta} g(\eta',\tau)d\eta', \qquad N(\xi,\tau)=\int^{\xi} f(\xi',\tau)d\xi'. \tag{4.15}$$

At first glance, it is seen that the terms proportional to ξ and η cause secularity. In order to remove the secularities we must have

$$\frac{\partial f}{\partial \tau}+\frac{3}{2}f\frac{\partial f}{\partial \xi}+\frac{1}{6}\frac{\partial^3 f}{\partial \xi^3}=0, \tag{4.16}$$

$$\frac{\partial g}{\partial \tau}-\frac{3}{2}g\frac{\partial g}{\partial \eta}-\frac{1}{6}\frac{\partial^3 g}{\partial \eta^3}=0. \tag{4.17}$$

These are Korteweg-de Vries equations. The solution of equations (4.13) and (4.14) for ζ_1 and w_1 gives

$$\begin{aligned}\zeta_1 =&F_1(\xi,\tau)+G_1(\eta,\tau)+\frac{1}{4}M(\eta,\tau)\frac{\partial f}{\partial \xi}+\frac{1}{4}N(\xi,\tau)\frac{\partial g}{\partial \eta}\\ &+\frac{1}{8}\left(f^2+g^2\right)+\frac{1}{2}fg-\frac{1}{6}\left(\frac{\partial^2 f}{\partial \xi^2}+\frac{\partial^2 g}{\partial \eta^2}\right),\end{aligned} \tag{4.18}$$

$$\begin{aligned}w_1 =&F_1(\xi,\tau)-G_1(\eta,\tau)+\frac{1}{4}M(\eta,\tau)\frac{\partial f}{\partial \xi}-\frac{1}{4}N(\xi,\tau)\frac{\partial g}{\partial \eta}\\ &-\frac{1}{8}\left(f^2-g^2\right)+\frac{1}{6}\left(\frac{\partial^2 f}{\partial \xi^2}-\frac{\partial^2 g}{\partial \eta^2}\right).\end{aligned} \tag{4.19}$$

Su and Mirie [6] stated that, although the terms $M(\eta,\tau)\frac{\partial f}{\partial \xi}$ and $N(\xi,\tau)\frac{\partial g}{\partial \eta}$ in (4.18) and (4.19) do not cause any secularity at this order but they will cause secularity in the next order perturbation expansion. However, in what follows it will be shown that it is not the case.

In order to obtain the localized progressive wave solution for the KdV equations (4.16) and (4.17), we shall seek a solution to these equations in the following form

$$f=f(\zeta_+), \qquad \zeta_+=\alpha_1\left(\xi-\alpha_2\tau\right), \tag{4.20}$$

$$g=g(\zeta_-)\ , \qquad \zeta_-=\beta_1\left(\eta+\beta_2\tau\right), \tag{4.21}$$

where α_i and β_i are constants to be determined from the solutions. Introducing the expression $f(\zeta_+)$ and $g(\zeta_-)$ into equations (4.16) and (4.17) we obtain

$$
\begin{aligned}
-\alpha_2 f^{'} + \frac{3}{2} f f^{'} + \frac{1}{6}\alpha_1^2 f^{'''} &= 0, &(4.22)\\
\beta_2 g^{'} - \frac{3}{2} g g^{'} - \frac{1}{6}\beta_1^2 g^{'''} &= 0, &(4.23)
\end{aligned}
$$

where a prime denotes the differentiation of the corresponding quantity with respect to its argument. Since we are concerned with the localized progressive wave solution, i.e., $f(g)$ and its various order derivatives vanish as $\zeta_+(\zeta_-) \to \pm\infty$. Integrating equations (4.22) and (4.23) with respect to ζ_+ and ζ_-, respectively, and using the localization condition we obtain

$$
\begin{aligned}
-\alpha_2 f + \frac{3}{4} f^2 + \frac{1}{6}\alpha_1^2 f^{''} &= 0, &(4.24)\\
\beta_2 g - \frac{3}{4} g^2 - \frac{1}{6}\beta_1^2 g^{''} &= 0. &(4.25)
\end{aligned}
$$

Here it is known that the equations (4.24) and (4.25) admit the progressive wave solution of the forms

$$
\begin{aligned}
f &= A\,\mathrm{sech}^2\,\zeta_+ \ , &(4.26)\\
g &= B\,\mathrm{sech}^2\,\zeta_- \ , &(4.27)
\end{aligned}
$$

where A and B are the amplitudes of the solitary waves and the other quantities are defined by

$$
\begin{aligned}
\alpha_1 &= \left(\frac{3A}{4}\right)^{1/2}, \quad \alpha_2 = \frac{A}{2},\\
\beta_1 &= \left(\frac{3B}{4}\right)^{1/2}, \quad \beta_2 = \frac{B}{2}. &(4.28)
\end{aligned}
$$

Substituting (4.10), (4.18) and (4.19) into the set of equations (4.9), we obtain

$$
\begin{aligned}
&2\frac{\partial}{\partial\eta}(\zeta_2 + w_2) + 2\frac{\partial F_1}{\partial\tau} + 3\frac{\partial}{\partial\xi}(fF_1) + \frac{1}{3}\frac{\partial^3 F_1}{\partial\xi^3} - \frac{3}{8}f^2\frac{\partial f}{\partial\xi} + \frac{1}{12}f\frac{\partial^3 f}{\partial\xi^3}\\
&+ \frac{11}{12}\frac{\partial f}{\partial\xi}\frac{\partial^2 f}{\partial\xi^2} - 2\frac{dp_0}{d\tau}\frac{\partial f}{\partial\xi} + \frac{1}{45}\frac{\partial^5 f}{\partial\xi^5} - \frac{1}{2}\frac{\partial^3 f}{\partial\tau\partial\xi^2} - f\frac{\partial G_1}{\partial\eta} - \frac{\partial f}{\partial\xi}G_1\\
&- \frac{\partial}{\partial\eta}(gG_1) + \frac{2}{3}\frac{\partial^3 G_1}{\partial\eta^3} - F_1\frac{\partial g}{\partial\eta} - g\frac{\partial F_1}{\partial\xi} - \frac{1}{4}\frac{\partial^2 f}{\partial\xi^2}gM - \frac{1}{4}\frac{\partial f}{\partial\xi}\frac{\partial g}{\partial\eta}M
\end{aligned}
$$

$$+\left(\frac{1}{6}\frac{\partial^4 g}{\partial\eta^4}-\frac{1}{4}\frac{\partial}{\partial\eta}\left(g\frac{\partial g}{\partial\eta}\right)-\frac{1}{4}f\frac{\partial^2 g}{\partial\eta^2}-\frac{1}{4}\frac{\partial f}{\partial\xi}\frac{\partial g}{\partial\eta}\right)N+\frac{1}{4}fg\frac{\partial g}{\partial\eta}$$
$$+\left(\frac{1}{8}f^2-\frac{1}{6}\frac{\partial^2 f}{\partial\xi^2}\right)\frac{\partial g}{\partial\eta}+\frac{11}{6}f\frac{\partial^3 g}{\partial\eta^3}+\left(\frac{3}{4}f\frac{\partial f}{\partial\xi}+\frac{13}{12}\frac{\partial^3 f}{\partial\xi^3}\right)g$$
$$+\frac{5}{12}\frac{\partial f}{\partial\xi}\frac{\partial^2 g}{\partial\eta^2}+\frac{3}{8}g^2\frac{\partial g}{\partial\eta}-4\frac{\partial P_0}{\partial\eta}\frac{\partial f}{\partial\xi}+\frac{29}{12}\frac{\partial g}{\partial\eta}\frac{\partial^2 g}{\partial\eta^2}+\frac{4}{3}g\frac{\partial^3 g}{\partial\eta^3}$$
$$+\frac{4}{45}\frac{\partial^5 g}{\partial\eta^5}=0, \tag{4.29}$$

$$2\frac{\partial}{\partial\xi}(\zeta_2-w_2)-2\frac{\partial G_1}{\partial\tau}+3\frac{\partial}{\partial\eta}(gG_1)+\frac{1}{3}\frac{\partial^3 G_1}{\partial\eta^3}-\frac{3}{8}g^2\frac{\partial g}{\partial\eta}+\frac{1}{12}g\frac{\partial^3 g}{\partial\eta^3}$$
$$+\frac{11}{12}\frac{\partial g}{\partial\eta}\frac{\partial^2 g}{\partial\eta^2}+2\frac{dq_0}{d\tau}\frac{\partial g}{\partial\eta}+\frac{1}{45}\frac{\partial^5 g}{\partial\eta^5}+\frac{1}{2}\frac{\partial^3 g}{\partial\tau\partial\eta^2}-g\frac{\partial F_1}{\partial\xi}-F_1\frac{\partial g}{\partial\eta}$$
$$-\frac{\partial}{\partial\xi}(fF_1)+\frac{2}{3}\frac{\partial^3 F_1}{\partial\xi^3}-\frac{\partial f}{\partial\xi}G_1-f\frac{\partial G_1}{\partial\eta}-\frac{1}{4}\frac{\partial g}{\partial\eta}fN-\frac{1}{4}\frac{\partial g}{\partial\eta}\frac{\partial f}{\partial\xi}N$$
$$+\left(\frac{1}{6}\frac{\partial^4 f}{\partial\xi^4}-\frac{1}{4}\frac{\partial}{\partial\xi}\left(f\frac{\partial f}{\partial\xi}\right)-\frac{1}{4}g\frac{\partial^2 f}{\partial\xi^2}-\frac{1}{4}\frac{\partial f}{\partial\xi}\frac{\partial g}{\partial\eta}\right)M+\frac{1}{4}gf\frac{\partial f}{\partial\xi}$$
$$+\left(\frac{1}{8}g^2-\frac{1}{6}\frac{\partial^2 g}{\partial\eta^2}\right)\frac{\partial f}{\partial\xi}+\frac{11}{6}g\frac{\partial^3 f}{\partial\xi^3}+\left(\frac{3}{4}g\frac{\partial g}{\partial\eta}+\frac{13}{12}\frac{\partial^3 g}{\partial\eta^3}\right)f$$
$$+\frac{5}{12}\frac{\partial g}{\partial\eta}\frac{\partial^2 f}{\partial\xi^2}+\frac{3}{8}f^2\frac{\partial f}{\partial\xi}-4\frac{\partial Q_0}{\partial\xi}\frac{\partial g}{\partial\eta}+\frac{29}{12}\frac{\partial f}{\partial\xi}\frac{\partial^2 f}{\partial\xi^2}+\frac{4}{3}f\frac{\partial^3 f}{\partial\xi^3}$$
$$+\frac{4}{45}\frac{\partial^5 f}{\partial\xi^5}=0. \tag{4.30}$$

Integrating (4.29) with respect to η and (4.30) with respect to ξ we obtain

$$2(\zeta_2+w_2)+\eta\left(2\frac{\partial F_1}{\partial\tau}+3\frac{\partial}{\partial\xi}(fF_1)+\frac{1}{3}\frac{\partial^3 F_1}{\partial\xi^3}-\frac{3}{8}f^2\frac{\partial f}{\partial\xi}+\frac{1}{12}f\frac{\partial^3 f}{\partial\xi^3}\right.$$
$$\left.+\frac{11}{12}\frac{\partial f}{\partial\xi}\frac{\partial^2 f}{\partial\xi^2}-2\frac{dp_0}{d\tau}\frac{\partial f}{\partial\xi}+\frac{1}{45}\frac{\partial^5 f}{\partial\xi^5}-\frac{1}{2}\frac{\partial^3 f}{\partial\tau\partial\xi^2}\right)-(f+g)G_1-gF_1$$
$$-\frac{\partial f}{\partial\xi}\int^{\eta}G_1 d\eta'+\frac{2}{3}\frac{\partial^2 G_1}{\partial\eta^2}+\left(\frac{3}{4}f\frac{\partial f}{\partial\xi}+\frac{13}{12}\frac{\partial^3 f}{\partial\xi^3}-\frac{\partial F_1}{\partial\xi}\right)M(\eta,\tau)$$
$$+\left(\frac{1}{6}\frac{\partial^3 g}{\partial\eta^3}-\frac{1}{4}f\frac{\partial g}{\partial\eta}-\frac{1}{4}g\frac{\partial f}{\partial\xi}-\frac{1}{4}g\frac{\partial g}{\partial\eta}\right)N(\xi,\tau)-\frac{1}{4}\frac{\partial^2 f}{\partial\xi^2}\int^{\eta}gM\,d\eta'$$
$$-\frac{1}{4}\frac{\partial f}{\partial\xi}\int^{\eta}\left(\frac{\partial g}{\partial\eta}M\right)d\eta'+\frac{1}{8}fg^2+\frac{1}{8}f^2g-\frac{1}{6}\frac{\partial^2 f}{\partial\xi^2}g+\frac{5}{12}\frac{\partial f}{\partial\xi}\frac{\partial g}{\partial\eta}$$

$$
\begin{aligned}
&+\frac{1}{8}g^3+\frac{11}{6}f\frac{\partial^2 g}{\partial\eta^2}+\frac{4}{3}g\frac{\partial^2 g}{\partial\eta^2}+\frac{13}{24}\left(\frac{\partial g}{\partial\eta}\right)^2+\frac{4}{45}\frac{\partial^4 g}{\partial\eta^4}\\
&-4P_0\frac{\partial f}{\partial\xi}=4F_2(\xi,\tau),
\end{aligned} \tag{4.31}
$$

$$
\begin{aligned}
&2(\zeta_2-w_2)+\xi\left(-2\frac{\partial G_1}{\partial\tau}+3\frac{\partial}{\partial\eta}(gG_1)+\frac{1}{3}\frac{\partial^3 G_1}{\partial\eta^3}-\frac{3}{8}g^2\frac{\partial g}{\partial\eta}+\frac{1}{12}g\frac{\partial^3 g}{\partial\eta^3}\right.\\
&\left.+\frac{11}{12}\frac{\partial g}{\partial\eta}\frac{\partial^2 g}{\partial\eta^2}+2\frac{dq_0}{d\tau}\frac{\partial g}{\partial\eta}+\frac{1}{45}\frac{\partial^5 g}{\partial\eta^5}+\frac{1}{2}\frac{\partial^3 g}{\partial\tau\partial\eta^2}\right)-(f+g)F_1-fG_1\\
&-\frac{\partial g}{\partial\eta}\int^{\xi}F_1 d\xi'+\frac{2}{3}\frac{\partial^2 F_1}{\partial\xi^2}+\left(\frac{3}{4}g\frac{\partial g}{\partial\eta}+\frac{13}{12}\frac{\partial^3 g}{\partial\eta^3}-\frac{\partial G_1}{\partial\eta}\right)N(\xi,\tau)\\
&+\left(\frac{1}{6}\frac{\partial^3 f}{\partial\xi^3}-\frac{1}{4}g\frac{\partial f}{\partial\xi}-\frac{1}{4}f\frac{\partial g}{\partial\eta}-\frac{1}{4}f\frac{\partial f}{\partial\xi}\right)M(\eta,\tau)-\frac{1}{4}\frac{\partial^2 g}{\partial\eta^2}\int^{\xi}fN d\xi'\\
&-\frac{1}{4}\frac{\partial g}{\partial\eta}\int^{\xi}\left(\frac{\partial f}{\partial\xi}N\right)d\xi'+\frac{1}{8}fg^2+\frac{1}{8}f^2 g+\frac{11}{6}\frac{\partial^2 f}{\partial\xi^2}g+\frac{5}{12}\frac{\partial f}{\partial\xi}\frac{\partial g}{\partial\eta}\\
&+\frac{1}{8}f^3-\frac{1}{6}f\frac{\partial^2 g}{\partial\eta^2}+\frac{4}{3}f\frac{\partial^2 f}{\partial\xi^2}+\frac{13}{24}\left(\frac{\partial f}{\partial\xi}\right)^2+\frac{4}{45}\frac{\partial^4 f}{\partial\xi^4}\\
&-4Q_0\frac{\partial g}{\partial\eta}=4G_2(\eta,\tau),
\end{aligned} \tag{4.32}
$$

where $F_2(\xi,\tau)$ and $G_2(\eta,\tau)$ are two unknown functions whose evolution equations will be obtained from next order equations. Again the terms proportional to ξ and η in these equations cause the secularity in the solution. In order to remove the secularity, the coefficient of η in (4.31) and the coefficient of ξ in (4.32) must vanish, that is

$$
\begin{aligned}
\frac{\partial F_1}{\partial\tau}+\frac{3}{2}\frac{\partial}{\partial\xi}(fF_1)+\frac{1}{6}\frac{\partial^3 F_1}{\partial\xi^3}=&\frac{3}{16}f^2\frac{\partial f}{\partial\xi}-\frac{1}{24}f\frac{\partial^3 f}{\partial\xi^3}-\frac{11}{24}\frac{\partial f}{\partial\xi}\frac{\partial^2 f}{\partial\xi^2}\\
&+\frac{dp_0}{d\tau}\frac{\partial f}{\partial\xi}-\frac{1}{90}\frac{\partial^5 f}{\partial\xi^5}+\frac{1}{4}\frac{\partial^3 f}{\partial\tau\partial\xi^2},
\end{aligned} \tag{4.33}
$$

$$
\begin{aligned}
\frac{\partial G_1}{\partial\tau}-\frac{3}{2}\frac{\partial}{\partial\eta}(gG_1)-\frac{1}{6}\frac{\partial^3 G_1}{\partial\eta^3}=&-\frac{3}{16}g^2\frac{\partial g}{\partial\eta}+\frac{1}{24}g\frac{\partial^3 g}{\partial\eta^3}+\frac{11}{24}\frac{\partial g}{\partial\eta}\frac{\partial^2 g}{\partial\eta^2}\\
&+\frac{dq_0}{d\tau}\frac{\partial g}{\partial\eta}+\frac{1}{90}\frac{\partial^5 g}{\partial\eta^5}+\frac{1}{4}\frac{\partial^3 g}{\partial\tau\partial\eta^2}.
\end{aligned} \tag{4.34}
$$

Noting the identities

$$\frac{\partial^2 f}{\partial \xi^2} = 3Af - \frac{9}{2}f^2, \quad \left(\frac{\partial f}{\partial \xi}\right)^2 = 3Af^2 - 3f^3,$$
$$\frac{\partial^4 f}{\partial \xi^4} = \frac{135}{2}f^3 - \frac{135}{2}Af^2 + 9A^2 f,$$
$$\frac{\partial^6 f}{\partial \xi^6} = \frac{-8505}{4}f^4 + 2835Af^3 - \frac{1701}{2}A^2 f^2 + 27A^3 f, \tag{4.35}$$

and keeping in mind that the similar identities for the derivatives of the function g, the equations (4.33) and (4.34) can be written as (Demiray [72])

$$\frac{\partial F_1}{\partial \tau} + \frac{3}{2}\frac{\partial}{\partial \xi}(fF_1) + \frac{1}{6}\frac{\partial^3 F_1}{\partial \xi^3} = \frac{\partial S_1(f)}{\partial \xi}, \tag{4.36}$$
$$\frac{\partial G_1}{\partial \tau} - \frac{3}{2}\frac{\partial}{\partial \eta}(gG_1) - \frac{1}{6}\frac{\partial^3 G_1}{\partial \eta^3} = \frac{\partial T_1(f)}{\partial \xi}, \tag{4.37}$$

where $S_1(f)$ and $T_1(f)$ are defined as follows

$$S_1(f) = \left(\frac{dp_0}{d\tau} - \frac{19A^2}{40}\right) f + \frac{9}{16}Af^2 + \frac{1}{8}f^3, \tag{4.38}$$
$$T_1(f) = \left(\frac{dq_0}{d\tau} + \frac{19B^2}{40}\right) g - \frac{9}{16}Bg^2 - \frac{1}{8}g^3. \tag{4.39}$$

As is seen from the equations (4.31) and (4.32) the other terms in the expression of ζ_2 and w_2 do not cause any secularity for this order, but it might be possible to have secularities in the next order. Seeking a progressive wave solution for the equations (4.36)-(4.39) of the form $F_1 = F_1(\zeta_+)$, $G_1 = G_1(\zeta_-)$, the following equations are obtained

$$\frac{A}{8}F_1''' + \frac{3}{2}(fF_1)' - \frac{A}{2}F_1' = S_1'(f), \tag{4.40}$$
$$-\frac{B}{8}G_1''' - \frac{3}{2}(gG_1)' + \frac{B}{2}G_1' = T_1'(f). \tag{4.41}$$

Integrating these equations with respect to ζ_+ and ζ_-, respectively, and using the localization condition, we obtain

$$\frac{A}{8}F_1'' + \frac{1}{2}(3f - A)F_1 = \left(\frac{dp_0}{d\tau} - \frac{19A^2}{40}\right) f + \frac{9}{16}Af^2 + \frac{1}{8}f^3, \tag{4.42}$$

$$-\frac{B}{8}G_1^{''}+\frac{1}{2}\left(B-3g\right)G_1=\left(\frac{dq_0}{d\tau}+\frac{19B^2}{40}\right)g-\frac{9}{16}Bg^2-\frac{1}{8}g^3. \tag{4.43}$$

The first terms in the right-hand side cause the secularity (see Demiray [71, 72]); therefore the coefficients of f and g must vanish, which yields

$$p_0=\frac{19}{40}A^2\tau, \qquad q_0=-\frac{19}{40}B^2\tau. \tag{4.44}$$

We shall propose a solution for F_1 and G_1 of the following form

$$F_1=a_1\operatorname{sech}^4\zeta_+ + a_2\operatorname{sech}^2\zeta_+, \tag{4.45}$$

$$G_1=b_1\operatorname{sech}^4\zeta_- + b_2\operatorname{sech}^2\zeta_-, \tag{4.46}$$

where a_i and b_i are constants to be determined from the solutions of (4.42) and (4.43), respectively. Carrying out the derivative of F_1 and G_1 we have

$$F_1^{''}=-20a_1\operatorname{sech}^6\zeta_+ + (16a_1-6a_2)\operatorname{sech}^4\zeta_+ + 4a_2\operatorname{sech}^2\zeta_+, \tag{4.47}$$

$$G_1^{''}=-20b_1\operatorname{sech}^6\zeta_- + (16b_1-6b_2)\operatorname{sech}^4\zeta_- + 4b_2\operatorname{sech}^2\zeta_-. \tag{4.48}$$

Inserting (4.45) and (4.47) into (4.42) and inserting (4.46) and (4.48) into (4.43) and setting the coefficients of $\operatorname{sech}^6\zeta_+$ and $\operatorname{sech}^4\zeta_+$ equal to zero, one has

$$a_1=-\frac{A^2}{8}, \quad a_2=A^2, \tag{4.49}$$

$$b_1=-\frac{B^2}{8}, \quad b_2=B^2. \tag{4.50}$$

Then the particular solution of the differential equations (4.42) and (4.43) may be given by

$$F_1=Af-\frac{1}{8}f^2, \qquad G_1=Bg-\frac{1}{8}g^2. \tag{4.51}$$

By using the above results one can obtain the following identities for the terms involving the functions g, G_1 and M

$$\int g^2 d\eta^{'}=\frac{M}{3}\left(g+2B\right), \quad \int G_1 d\eta^{'}=\frac{M}{24}(22B-g),$$

$$\int gM d\eta^{'}=-\frac{2}{3}g, \quad \int\left(\frac{\partial g}{\partial\eta}M\right)d\eta^{'}=\frac{2M}{3}(g-B). \tag{4.52}$$

Similar expressions are valid for the terms involving f, F_1 and N. Then the equations (4.31) and (4.32) may be written in the following form

$$\begin{aligned}\zeta_2 + w_2 =& \frac{1}{16}g^3 + \frac{43}{16}Bg^2 - \frac{7}{5}B^2 g + \left(\frac{A}{2} - \frac{9B}{4}\right) fg + 4fg^2 - \frac{1}{8}f^2 g \\ & - \frac{5}{24}\frac{\partial f}{\partial \xi}\frac{\partial g}{\partial \eta} + \left[-\frac{B}{4}\frac{\partial g}{\partial \eta} + \frac{7}{8}g\frac{\partial g}{\partial \eta} + \frac{1}{8}f\frac{\partial g}{\partial \eta} + \frac{1}{8}\frac{\partial f}{\partial \xi}g\right] N \\ & \left[\left(-\frac{9A}{8} + \frac{3B}{8}\right)\frac{\partial f}{\partial \xi} + \frac{35}{8}f\frac{\partial f}{\partial \xi} + \frac{1}{16}\frac{\partial f}{\partial \xi}g\right] M + 2P_0\frac{\partial f}{\partial \xi} \\ & + 2F_2(\xi,\tau), \end{aligned} \tag{4.53}$$

$$\begin{aligned}\zeta_2 - w_2 =& \frac{1}{16}f^3 + \frac{43}{16}Af^2 - \frac{7}{5}A^2 f + \left(\frac{B}{2} - \frac{9A}{4}\right) fg + 4f^2 g - \frac{1}{8}fg^2 \\ & - \frac{5}{24}\frac{\partial f}{\partial \xi}\frac{\partial g}{\partial \eta} + \left[-\frac{A}{4}\frac{\partial f}{\partial \xi} + \frac{7}{8}f\frac{\partial f}{\partial \xi} + \frac{1}{8}g\frac{\partial f}{\partial \xi} + \frac{1}{8}f\frac{\partial g}{\partial \eta}\right] M \\ & \left[\left(-\frac{9B}{8} + \frac{3A}{8}\right)\frac{\partial g}{\partial \eta} + \frac{35}{8}g\frac{\partial g}{\partial \eta} + \frac{1}{16}f\frac{\partial g}{\partial \eta}\right] N + 2Q_0\frac{\partial g}{\partial \eta} \\ & + 2G_2(\eta,\tau). \end{aligned} \tag{4.54}$$

In obtaining the equations (4.53) and (4.54) we have utilized the identities (4.35) and the similar identities that are valid for the function g.

As might be seen from equations (4.53) and (4.54) the terms appearing in the expressions of ζ_1 and w_1 do not cause any secularity in the solution of ζ_2 and w_2. Therefore the statement by Su and Mirie [6] is incorrect. However as we stated before, some of the terms appearing in the expressions of ζ_2 and w_2 (the equations (4.53) and (4.54)) may cause additional secularity in the expressions of ζ_3 and w_3. There appears to be two types of secularity in the solution of $\mathcal{O}(\epsilon^4)$ equation. As was seen before, the first type of secularity results from the terms proportional to ξ and η which will be studied later. The second type secularity occurs from the terms proportional $\int^{\xi} N(\xi^{'},\tau)d\xi^{'}$ and $\int^{\eta} M(\eta^{'},\tau)d\eta^{'}$ as $\xi(\eta) \to \pm\infty$. Here we shall first only consider the parts of $\mathcal{O}(\epsilon^4)$ equations leading to $\int^{\eta} M(\eta^{'},\tau)d\eta^{'}$ type of secularity. Similar expressions may be valid for $\int^{\xi} N(\xi^{'},\tau)d\xi^{'}$ type of secularity.

For this purpose we consider the following part of the $\mathcal{O}(\epsilon^4)$ equation

$$\begin{aligned}
&2\frac{\partial}{\partial\eta}(\zeta_3+w_3)+\frac{\partial}{\partial\tau}(\zeta_2+w_2)+\frac{3}{4}\frac{\partial}{\partial\xi}\left[(\zeta_0+w_0)(\zeta_2+w_2)\right]\\
&+\frac{1}{6}\frac{\partial^3}{\partial\xi^3}(\zeta_2+w_2)-\frac{dp_0}{d\tau}\frac{\partial}{\partial\xi}(\zeta_1+w_1)+\frac{1}{2}\frac{\partial^2 w_0}{\partial\xi^2}\frac{\partial}{\partial\xi}(\zeta_1+w_1)\\
&+\frac{1}{2}\frac{\partial}{\partial\xi}(\zeta_0+w_0)\frac{\partial^2 w_1}{\partial\xi^2}-\frac{1}{30}\frac{\partial^5 w_1}{\partial\xi^5}-\frac{1}{2}\frac{\partial^3 w_1}{\partial\tau\partial\xi^2}+w_1\frac{\partial}{\partial\xi}(\zeta_1+w_1)\\
&+\zeta_1\frac{\partial w_1}{\partial\xi}-\frac{1}{2}\frac{\partial}{\partial\xi}\left[\zeta_0(\zeta_2-w_2)\right]-\frac{1}{6}\frac{\partial^3}{\partial\xi^3}(\zeta_2-w_2)=0. \qquad (4.55)
\end{aligned}$$

A similar expression may be given for $2\frac{\partial}{\partial\xi}(\zeta_3-w_3)$ equation. We split (4.55) into two parts which contain the variables ζ_2+w_2 and $(\zeta_1,\ w_1,\ \zeta_2-w_2)$, respectively. Then, we obtain:

$$\begin{aligned}
&\frac{\partial}{\partial\tau}(\zeta_2+w_2)+\frac{3}{4}\frac{\partial}{\partial\xi}\left[(\zeta_0+w_0)(\zeta_2+w_2)\right]+\frac{1}{6}\frac{\partial^3}{\partial\xi^3}(\zeta_2+w_2)=\\
&\frac{35}{16}\left[\frac{189}{4}f^4-63Af^3+18A^2f^2\right]M, \qquad (4.56)
\end{aligned}$$

$$\begin{aligned}
&\frac{1}{2}\frac{\partial^2 w_0}{\partial\xi^2}\frac{\partial}{\partial\xi}(\zeta_1+w_1)-\frac{dp_0}{d\tau}\frac{\partial}{\partial\xi}(\zeta_1+w_1)+\frac{1}{2}\frac{\partial}{\partial\xi}(\zeta_0+w_0)\frac{\partial^2 w_1}{\partial\xi^2}\\
&-\frac{1}{30}\frac{\partial^5 w_1}{\partial\xi^5}-\frac{1}{2}\frac{\partial^3 w_1}{\partial\tau\partial\xi^2}+w_1\frac{\partial}{\partial\xi}(\zeta_1+w_1)+\zeta_1\frac{\partial w_1}{\partial\xi}\\
&-\frac{1}{2}\frac{\partial}{\partial\xi}\left[\zeta_0(\zeta_2-w_2)\right]-\frac{1}{6}\frac{\partial^3}{\partial\xi^3}(\zeta_2-w_2)=\\
&\frac{1}{16}\left[\frac{189}{4}f^4-63Af^3+18A^2f^2\right]M, \qquad (4.57)
\end{aligned}$$

where we have used the identities given by (4.35) and (4.52). As is seen from the equations (4.56) and (4.57), the terms proportional to $M(\eta,\tau)$ do not vanish and they cause the secularity of the type $\int^{\eta} M(\eta',\tau)d\eta'$ in the expression of ζ_3 and w_3. Similar expression may be given for $\int^{\xi} N(\xi',\tau)d\xi'$ type of secularities.

By direct substitution in the expressions of $\zeta_2 + w_2$ and $\zeta_2 - w_2$

$$P_0 = -\frac{9}{4} f(\xi,\tau) M(\eta,\tau), \qquad Q_0 = -\frac{9}{4} g(\eta,\tau) N(\xi,\tau), \tag{4.58}$$

these secularities may be removed. These expressions make it possible to determine phase shift functions.

To obtain the secularities of type η (or ξ) we use the following part of the $\mathcal{O}(\epsilon^4)$ equation to obtain the governing equation for $F_2(\xi,\tau)$

$$\begin{aligned}
&2\frac{\partial}{\partial\eta}(\zeta_3 + w_3) + \frac{\partial}{\partial\tau}(\zeta_2 + w_2) + w_0\frac{\partial}{\partial\xi}(\zeta_2 + w_2) + \frac{\partial w_0}{\partial\xi}(\zeta_2 + w_2)\\
&+ \frac{\partial}{\partial\xi}(\zeta_0 w_2) + \frac{1}{3}\frac{\partial^3 w_2}{\partial\xi^3} - \frac{dp_0}{d\tau}\frac{\partial}{\partial\xi}(\zeta_1 + w_1) + \frac{1}{2}\frac{\partial}{\partial\xi}\left[\frac{\partial^2 w_0}{\partial\xi^2}\zeta_1\right]\\
&+ \frac{\partial}{\partial\xi}(\zeta_1 w_1) + \frac{1}{2}\frac{\partial}{\partial\xi}\left[\frac{\partial w_0}{\partial\xi}\frac{\partial w_1}{\partial\xi}\right] - \frac{1}{30}\frac{\partial^5 w_1}{\partial\xi^5} + \frac{1}{2}\frac{\partial}{\partial\xi}\left[\zeta_0\frac{\partial^2 w_1}{\partial\xi^2}\right]\\
&- \frac{1}{2}\frac{\partial^3 w_1}{\partial\xi^2\partial\tau} - \frac{1}{2}w_0\frac{\partial^3 w_1}{\partial\xi^3} - \frac{1}{2}\frac{\partial^3 w_0}{\partial\xi^3}w_1 + w_1\frac{\partial w_1}{\partial\xi} - \frac{dp_1}{d\tau}\frac{\partial}{\partial\xi}(\zeta_0 + w_0)\\
&+ \left(\frac{\partial w_0}{\partial\xi}\right)^2\frac{\partial\zeta_0}{\partial\xi} - \frac{1}{8}\frac{\partial^4 w_0}{\partial\xi^4}\frac{\partial\zeta_0}{\partial\xi} - \frac{\partial^2 w_0}{\partial\xi\partial\tau}\frac{\partial\zeta_0}{\partial\xi} - w_0\frac{\partial^2 w_0}{\partial\xi^2}\frac{\partial\zeta_0}{\partial\xi}\\
&+ \frac{\partial w_0}{\partial\xi}\frac{\partial^2 w_0}{\partial\xi^2}\zeta_0 - \frac{1}{8}\frac{\partial^5 w_0}{\partial\xi^5}\zeta_0 - \frac{\partial^3 w_0}{\partial\xi^2\partial\tau}\zeta_0 - w_0\frac{\partial^3 w_0}{\partial\xi^3}\zeta_0 + \frac{1}{2}\frac{dp_0}{d\tau}\frac{\partial^3 w_0}{\partial\xi^3}\\
&+ \frac{1}{12}\frac{\partial^2 w_0}{\partial\xi^2}\frac{\partial^3 w_0}{\partial\xi^3} - \frac{1}{8}\frac{\partial w_0}{\partial\xi}\frac{\partial^4 w_0}{\partial\xi^4} + \frac{1}{840}\frac{\partial^7 w_0}{\partial\xi^7} + \frac{1}{24}\frac{\partial^5 w_0}{\partial\xi^4\partial\tau}\\
&+ \frac{1}{24}w_0\frac{\partial^5 w_0}{\partial\xi^5} = 0.
\end{aligned} \tag{4.59}$$

We substitute the field variables into (4.59) then the terms proportional to η in this equation cause the secularity. In order to remove secularity, the coefficient of η in (4.59) must vanish, that is

$$\frac{\partial F_2}{\partial\tau} + \frac{3}{2}\frac{\partial}{\partial\xi}(fF_2) + \frac{1}{6}\frac{\partial^3 F_2}{\partial\xi^3} = R(\xi,\tau) \tag{4.60}$$

where $R(\xi,\tau)$ is defined as follows

$$\begin{aligned}
R(\xi,\tau) =& \frac{1}{4}\frac{\partial^3 F_1}{\partial\tau\partial\xi^2} - \frac{1}{90}\frac{\partial^5 F_1}{\partial\xi^5} - \frac{1}{24}\frac{\partial}{\partial\xi}\left(f\frac{\partial^2 F_1}{\partial\xi^2}\right) - \frac{1}{24}\frac{\partial}{\partial\xi}\left(\frac{\partial^2 f}{\partial\xi^2}F_1\right)\\
&- \frac{5}{12}\frac{\partial}{\partial\xi}\left(\frac{\partial f}{\partial\xi}\frac{\partial F_1}{\partial\xi}\right) + \frac{3}{16}\frac{\partial}{\partial\xi}(fF_1) - \frac{3}{2}F_1\frac{\partial F_1}{\partial\xi} + \frac{dp_0}{d\tau}\frac{\partial F_1}{\partial\xi}
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{48}\frac{\partial^5 f}{\partial\tau\partial\xi^4}+\frac{7}{16}f\frac{\partial^3 f}{\partial\tau\partial\xi^2}+\frac{3}{8}\frac{\partial f}{\partial\xi}\frac{\partial^2 f}{\partial\tau\partial\xi}-\frac{1}{16}\frac{\partial f}{\partial\tau}\frac{\partial^2 f}{\partial\xi^2}\\
&-\frac{23}{15120}\frac{\partial^7 f}{\partial\xi^7}-\frac{7}{240}f\frac{\partial^5 f}{\partial\xi^5}-\frac{97}{480}\frac{\partial f}{\partial\xi}\frac{\partial^4 f}{\partial\xi^4}-\frac{11}{32}\frac{\partial^2 f}{\partial\xi^2}\frac{\partial^3 f}{\partial\xi^3}\\
&-\frac{1}{4}\frac{dp_0}{d\tau}\frac{\partial^3 f}{\partial\xi^3}-\frac{B}{8}\frac{\partial f}{\partial\xi}\frac{\partial^2 f}{\partial\xi^2}+\frac{47}{192}f^2\frac{\partial^3 f}{\partial\xi^3}-\frac{11}{24}f\frac{\partial f}{\partial\xi}\frac{\partial^2 f}{\partial\xi^2}\\
&-\frac{91}{192}\left(\frac{\partial f}{\partial\xi}\right)^3+\frac{dp_1}{d\tau}\frac{\partial f}{\partial\xi}-\frac{3}{4}f^3\frac{\partial f}{\partial\xi}. \qquad (4.61)
\end{aligned}
$$

Seeking a progressive wave solution for the equation (4.60) of the form $F_2 = F_2(\zeta_+)$ and introducing the identities (4.35) into (4.61), the following equation is obtained

$$
\frac{A}{8}F_2^{'''}+\frac{3}{2}(fF_2)^{'}-\frac{A}{2}F_2^{'}=S^{'}(f), \qquad (4.62)
$$

where $S(f)$ is defined as follows

$$
\begin{aligned}
S(f)=&\left(\frac{dp_1}{d\tau}-\frac{55}{112}A^3\right)f-\left(\frac{393}{320}+\frac{3B}{16A}\right)A^2f^2\\
&+\left(\frac{201}{32}+\frac{3B}{16A}\right)Af^3-\frac{591}{128}f^4. \qquad (4.63)
\end{aligned}
$$

Integrating this equation with respect to ζ_+ and using localization condition, we have

$$
\begin{aligned}
\frac{A}{8}F_2^{''}+\frac{1}{2}(3f-A)F_2=&\left(\frac{dp_1}{d\tau}-\frac{55}{112}A^3\right)f-\left(\frac{393}{320}+\frac{3B}{16A}\right)A^2f^2\\
&+\left(\frac{201}{32}+\frac{3B}{16A}\right)Af^3-\frac{591}{128}f^4. \qquad (4.64)
\end{aligned}
$$

The first term in the right-hand side of the equation (4.64) causes the secularity; therefore the coefficient of f must vanish, that is,

$$
p_1(\tau)=\frac{55}{112}A^3\tau. \qquad (4.65)
$$

We shall propose a solution for F_2 of the following form

$$
F_2=c_1\operatorname{sech}^6\zeta_+ + c_2\operatorname{sech}^4\zeta_+ + c_3\operatorname{sech}^2\zeta_+ \qquad (4.66)
$$

where c_i are constants to be determined from the solution of (4.64). Taking the derivative of F_2 we obtain

$$
\begin{aligned}
F_2^{''} =& -42c_1 \operatorname{sech}^8\zeta_+ + (36c_1 - 20c_2)\operatorname{sech}^6\zeta_+ \\
&+ (16c_2 - 6c_3)\operatorname{sech}^4\zeta_+ + 4c_3\operatorname{sech}^2\zeta_+.
\end{aligned} \tag{4.67}
$$

Inserting (4.66) and (4.67) into (4.64) and setting the coefficients of $\operatorname{sech}^8\zeta_+$, $\operatorname{sech}^6\zeta_+$ and $\operatorname{sech}^4\zeta_+$ equal to zero, we have

$$
c_1 = \frac{197}{160}A^3, \quad c_2 = -\left(\frac{217}{160}A^3 + \frac{3A^2B}{16}\right), \quad c_3 = \frac{43}{40}A^3 + \frac{A^2B}{8}. \tag{4.68}
$$

Then the particular solution of the differential equation (4.64) can be written as

$$
F_2 = \frac{197}{160}f^3 - \left(\frac{217}{160} + \frac{3B}{16A}\right)Af^2 + \left(\frac{43}{40} + \frac{B}{8A}\right)A^2 f, \tag{4.69}
$$

Similarly, for other unknowns G_2 and q_1 we have

$$
\begin{aligned}
G_2 &= \frac{197}{160}g^3 - \left(\frac{217}{160} + \frac{3A}{16B}\right)Bg^2 + \left(\frac{43}{40} + \frac{A}{8B}\right)B^2 g, \\
q_1(\tau) &= -\frac{55}{112}B^3\tau.
\end{aligned} \tag{4.70}
$$

Then, the final solution for ζ_2 and w_2 take the following form

$$
\begin{aligned}
\zeta_2 =& \frac{101}{80}(f^3 + g^3) + \frac{31}{16}(fg^2 + f^2 g) - \frac{1}{80}(Af^2 + Bg^2) \\
&- \frac{3}{16}(Bf^2 + Ag^2) - \frac{7}{8}(A+B)fg + \frac{3}{8}(A^2 f + B^2 g) \\
&+ \frac{1}{8}AB(f+g) - \frac{5}{24}\frac{\partial f}{\partial \xi}\frac{\partial g}{\partial \eta} + \left[\left(\frac{-11A + 3B}{16}\right)\frac{\partial f}{\partial \xi} + \frac{3}{8}f\frac{\partial f}{\partial \xi}\right. \\
&\left. + \frac{1}{16}f\frac{\partial g}{\partial \eta} + \frac{3}{32}\frac{\partial f}{\partial \xi}g\right]M + \left[\left(\frac{3A - 11B}{16}\right)\frac{\partial g}{\partial \eta} + \frac{3}{8}g\frac{\partial g}{\partial \eta}\right. \\
&\left. + \frac{1}{16}\frac{\partial f}{\partial \xi}g + \frac{3}{32}f\frac{\partial g}{\partial \eta}\right]N,
\end{aligned} \tag{4.71}
$$

$$
\begin{aligned}
w_2 =& \frac{6}{5}(f^3 - g^3) + \frac{33}{16}(fg^2 - f^2 g) - \frac{27}{10}(Af^2 - Bg^2) \\
&- \frac{3}{16}(Bf^2 - Ag^2) + \frac{11}{8}(A-B)fg + \frac{71}{40}(A^2 f - B^2 g)
\end{aligned}
$$

$$+\frac{1}{8}AB(f-g)+\left[\left(\frac{-7A+3B}{16}\right)\frac{\partial f}{\partial \xi}-\frac{1}{2}f\frac{\partial f}{\partial \xi}-\frac{1}{16}f\frac{\partial g}{\partial \eta}\right.$$
$$\left.-\frac{1}{32}\frac{\partial f}{\partial \xi}g\right]M+\left[\left(\frac{-3A+7B}{16}\right)\frac{\partial g}{\partial \eta}+\frac{1}{2}g\frac{\partial g}{\partial \eta}+\frac{1}{16}\frac{\partial f}{\partial \xi}g\right.$$
$$\left.+\frac{1}{32}f\frac{\partial g}{\partial \eta}\right]N. \tag{4.72}$$

Thus, for this order the trajectories of the solitary waves become

$$\epsilon^{\frac{1}{2}}(x-t)=\xi+\epsilon p_0+\epsilon^2(p_1+P_0)+\mathcal{O}(\epsilon^3),$$
$$\epsilon^{\frac{1}{2}}(x+t)=\eta+\epsilon q_0+\epsilon^2(q_1+Q_0)+\mathcal{O}(\epsilon^3). \tag{4.73}$$

4.4.1 Phase Shifts

To obtain the phase shifts after a head-on collision of solitary waves characterized by A and B are asymptotically far from each other at the initial time ($t=-\infty$), the solitary wave A is at $\xi=0$, $\eta=-\infty$, and the solitary wave B is at $\eta=0$, $\xi=+\infty$, respectively. After the collision ($t=+\infty$), the solitary wave B is far to the right of solitary wave A, i.e., the solitary wave A is at $\xi=0$, $\eta=+\infty$, and the solitary wave B is at $\eta=0$, $\xi=-\infty$. Using (4.58) and (4.73) one can obtain the corresponding phase shifts Δ_A and Δ_B as follows:

$$\Delta_A=\epsilon^{1/2}(x-t)\mid_{\xi=0,\eta=\infty}-\epsilon^{1/2}(x-t)\mid_{\xi=0,\eta=-\infty}$$
$$=-\epsilon^2\frac{9}{4}f(0)\int_{-\infty}^{+\infty}g(\eta^{'})d\eta^{'}$$
$$=-\epsilon^2\frac{9A}{4}\int_{-\infty}^{+\infty}g(\eta^{'})d\eta^{'}, \tag{4.74}$$

$$
\begin{aligned}
\Delta_B &= \epsilon^{1/2}(x+t)\mid_{\eta=0,\xi=-\infty} - \epsilon^{1/2}(x+t)\mid_{\eta=0,\xi=\infty} \\
&= \epsilon^2 \frac{9}{4} g(0) \int\limits_{-\infty}^{+\infty} f(\xi^{'})d\xi^{'} \\
&= \epsilon^2 \frac{9B}{4} \int\limits_{-\infty}^{+\infty} f(\xi^{'})d\xi^{'}. \qquad (4.75)
\end{aligned}
$$

Using the explicit expressions of $f(\xi)$ and $g(\eta)$ the phase shifts are obtained as

$$
\Delta_A = -\epsilon^2 3\sqrt{3} A B^{1/2}, \quad \Delta_B = \epsilon^2 3\sqrt{3} A^{1/2} B. \qquad (4.76)
$$

Here, as opposed to the results of previous works on the same subject the phase shifts depend on the amplitudes of both waves.

4.5 Summary of the results

In the previous section, we have obtained the following results

$$
\begin{aligned}
\hat{\zeta}(f,g) =& \epsilon \Bigg\{ (f+g) + \epsilon \left(\frac{3}{4}(f^2+g^2) + \frac{1}{2}(Af+Bg) + \frac{1}{2}fg \right. \\
&\left. + \frac{1}{4}M(\eta,\tau)\frac{\partial f}{\partial \xi} + \frac{1}{4}N(\xi,\tau)\frac{\partial g}{\partial \eta} \right) + \epsilon^2 \left(\frac{101}{80}(f^3+g^3) \right. \\
& - \frac{1}{80}(Af^2+Bg^2) - \frac{3}{16}(Bf^2+Ag^2) + \frac{3}{8}(A^2 f + B^2 g) \\
& + \frac{1}{8}AB(f+g) + \frac{31}{16}(fg^2+f^2 g) - \frac{7}{8}(A+B)fg \\
& - \frac{5}{24}\frac{\partial f}{\partial \xi}\frac{\partial g}{\partial \eta} + \left[\left(\frac{-11A+3B}{16} \right) \frac{\partial f}{\partial \xi} + \frac{3}{8} f \frac{\partial f}{\partial \xi} + \frac{1}{16} f \frac{\partial g}{\partial \eta} \right. \\
& \left. + \frac{3}{32}\frac{\partial f}{\partial \xi} g \right] M + \left[\left(\frac{3A-11B}{16} \right) \frac{\partial g}{\partial \eta} + \frac{3}{8} g \frac{\partial g}{\partial \eta} + \frac{1}{16}\frac{\partial f}{\partial \xi} g \right. \\
& \left.\left. + \frac{3}{32} f \frac{\partial g}{\partial \eta} \right] N \right) + \ldots \Bigg\}. \qquad (4.77)
\end{aligned}
$$

Similar expression may be given for $\hat{w}(f, g)$.

$$p(\tau) = \epsilon \left(\frac{19}{40} A^2 + \epsilon \frac{55}{112} A^3 \right) \tau, \tag{4.78}$$

$$q(\tau) = \epsilon \left(-\frac{19}{40} B^2 - \epsilon \frac{55}{112} B^3 \right) \tau, \tag{4.79}$$

$$P = -\frac{9}{4} f(\xi, \tau) \int_{-\infty}^{\eta} g(\eta', \tau) d\eta', \tag{4.80}$$

$$Q = -\frac{9}{4} g(\eta, \tau) \int_{\infty}^{\xi} f(\xi', \tau) d\xi', \tag{4.81}$$

and

$$\zeta_+ = \left(\frac{3A\epsilon}{4} \right)^{\frac{1}{2}} \left(x - c_R t + \frac{9}{4} \epsilon^{\frac{3}{2}} f(\xi, \tau) \int_{-\infty}^{\eta} g(\eta', \tau) d\eta' \right), \tag{4.82}$$

$$\zeta_- = \left(\frac{3B\epsilon}{4} \right)^{\frac{1}{2}} \left(x + c_L t + \frac{9}{4} \epsilon^{\frac{3}{2}} g(\eta, \tau) \int_{\infty}^{\xi} f(\xi', \tau) d\xi' \right), \tag{4.83}$$

where c_R and c_L are defined by

$$c_R = 1 + \left(\epsilon \frac{A}{2} + \epsilon^2 \frac{19}{40} A^2 + \epsilon^3 \frac{55}{112} A^3 \right), \tag{4.84}$$

$$c_L = 1 + \left(\epsilon \frac{B}{2} + \epsilon^2 \frac{19}{40} B^2 + \epsilon^3 \frac{55}{112} B^3 \right). \tag{4.85}$$

The equations (4.80) and (4.81) serve to define the phase changes. Before the collision

$$\eta \to -\infty, \quad P \to 0, \qquad \xi \to \infty, \quad Q \to 0 \tag{4.86}$$

and after the collision

$$\eta \to \infty, \qquad P = -9A \left(\frac{B}{3} \right)^{\frac{1}{2}} \operatorname{sech}^2 \zeta_+, \tag{4.87}$$

$$\xi \to -\infty, \qquad Q = 9B \left(\frac{A}{3} \right)^{\frac{1}{2}} \operatorname{sech}^2 \zeta_-. \tag{4.88}$$

In this section we shall illustrate the profiles of right-going waves before and

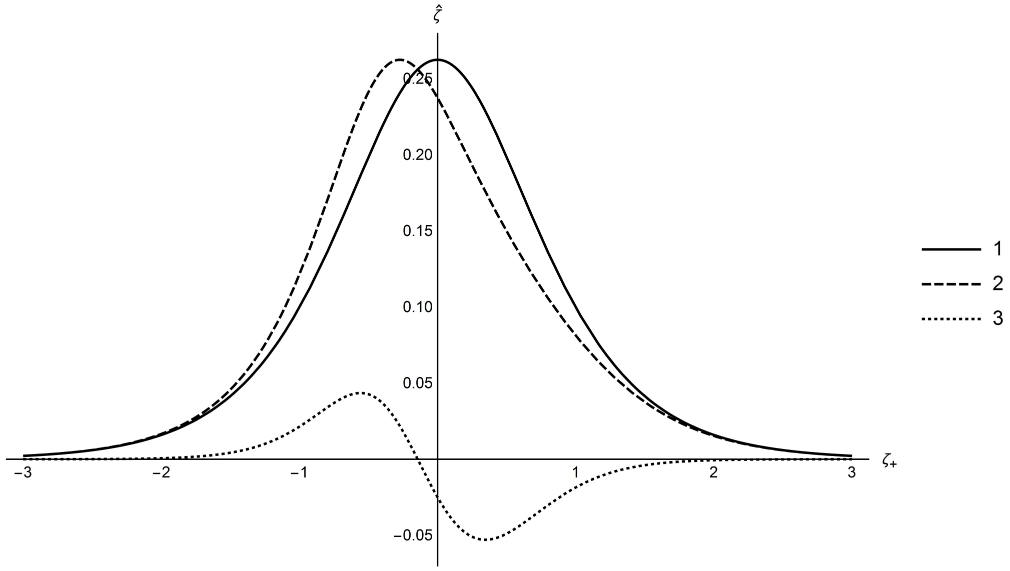

Figure 4.1: Right-going wave profile $\hat{\zeta}$ for $\epsilon = 0.4$, $A = B = 0.5$. 1: before collision; 2: after collision; 3: difference between the wave profiles before and after the collision.

after the collision. For that purpose we set $g(\eta, \tau) = 0$ in the expression $\hat{\zeta}$ and obtain

$$\hat{\zeta} = \epsilon \left\{ f + \epsilon \left(\frac{3}{4} f^2 + \frac{A}{2} f \right) + \epsilon^2 \left(\frac{101}{80} f^3 - \frac{A}{80} f^2 - \frac{3B}{16} f^2 + \frac{3A^2}{8} f \right. \right.$$
$$\left. \left. + \frac{AB}{8} f \right) + \ldots \right\} \tag{4.89}$$

with

$$f = A \operatorname{sech}^2 \left[\left(\frac{3A\epsilon}{4} \right)^{\frac{1}{2}} (x - c_R t + \Theta) \right] \tag{4.90}$$

where

$$\Theta = \epsilon^{\frac{3}{2}} 9A \left(\frac{B}{3} \right)^{\frac{1}{2}} \operatorname{sech}^2 \zeta_+ . \tag{4.91}$$

The variations of the wave profiles for surface elevation parameter $\hat{\zeta}$ before the collision ($\Theta = 0$) and after the collision (Θ is given as in (4.91)) are depicted in Figure 4.1, for various values of parameters ϵ, A and B. As is seen from the figure the wave profile before the collision is symmetric, whereas after the collision it is unsymmetrical and tilts backward with respect to the direction of its propagation.

4.6 Result and Conclusion

Utilizing the non-dimensionalized equations (4.1) and (4.2) and employing the extended PLK method, which is the combination of the classical reductive perturbation method and the strained coordinates, in order to obtain evolution equations as partial differential equations rather than ordinary differential equations we have studied the head-on collision of solitary waves in shallow water theory. Introducing a set of stretched coordinates that include some unknown functions which are to be determined from the removal of possible secularities in the solution, expanding these unknown functions and the field variables into power series of the smallness parameter ϵ and introducing the resulting expansions into the field equations we obtained the sets of partial differential equations. By solving these differential equations and imposing the requirements for the removal of possible secularities we obtained the speed correction terms and the trajectory functions. Our calculations show that the present results are exactly the same with those found in the previous chapter, whereas it is totally different from the results of Su and Mirie [6]. The variations of the wave profiles for right-going wave ($\hat{\zeta}$) before and after the collision are illustrated in Figure 4.1. As is seen from the figure the wave profile is symmetric before the collision whereas it is unsymmetrical after the collision with tilts backward with respect to the direction of its propagation.

Chapter 5

Head-on Collision of the Solitary Waves in Fluid-filled Elastic Tubes

5.1 Introduction

As was stated in Chapter 2, the pulsatile character of the blood wave is soliton-like and and it can be interpreted in terms of solitons. The solitary wave model gives a reasonable explanation for the peaking and steepening of pulsatile waves in arteries. The blood flow in arteries can be considered as an incompressible fluid flowing in a thin non-linear elastic tube, so head-on collision of solitary waves in fluid-filled elastic tubes had been studied by several researchers in this regard. In all of these studies the method proposed by Su and Mirie have been employed. Since the statement made by Su and Mirie is incorrect, as shown in Chapters 3 and 4, it is our motivation to study the head-on collision of solitary waves in arteries by employing the field equations for the fluid filled elastic tube and the extended PLK method. For that purpose, we introduce a set of stretched coordinates which include some unknown functions characterizing the higher order dispersive effects and the trajectory functions to be determined from the removal of possible secularities that might occur in the solution. Expanding these unknown functions and the field variables into power series of the smallness parameter ϵ and introducing the resulting expansions into the field equations we obtained the sets of partial differential equations governing the coefficients of the series. By solving these differential equations and imposing the non-secularity conditions in the solution we obtained various evolution equations. By seeking a progressive wave solution to these evolution equations we obtained the speed correction terms and the trajectory functions. The results of our calculation show that both the evolution equations and the phase shifts resulting from the head-on collision of solitary waves are quite different from those of Xue [7], who employed the incorrect formulation of Su and Mirie [6]. As opposed to the result of previous works on the same subject, in the present work the phase shifts depend on the amplitudes of both colliding waves. It is further observed that the order of the trajectory functions is ϵ^2, rather than ϵ. The variations of the wave profiles of right-going waves before and after the collision are depicted on Figure 5.1. It is seen that the wave profile before the collision is symmetric, whereas after the collision it is unsymmetrical and tilts backward with respect to the direction of its propagation.

5.2 Basic Equations

The equations (2.24)-(2.26) will be used as we study the head-on collision problem in fluid-filled elastic tubes. We can rewrite these equations as follows:

$$\frac{\partial S}{\partial t} + \frac{\partial u}{\partial x} + \frac{\partial}{\partial x}(Su) = 0, \tag{5.1}$$

$$\frac{\partial u}{\partial t} + \frac{\partial \pi}{\partial x} + \frac{\partial}{\partial x}\left(\frac{u^2}{2}\right) = 0, \tag{5.2}$$

$$\pi = \frac{2}{2+S}\frac{\partial^2 S}{\partial t^2} + \frac{2S(2+\alpha S)}{(2+S)^2}, \tag{5.3}$$

where x and t are the non-dimensional space and time parameters, S is the change in the cross-sectional area of the tube, u and π are the axial velocity and the pressure of the fluid body, respectively, and α characterizes the nonlinearity of the tube material.

5.3 Extended PLK Method

Motivated with the results found in Chapter 4, we introduce the following stretched coordinates

$$
\begin{aligned}
\epsilon^{\frac{1}{2}}(x-t) =&\xi + \epsilon p(\tau) + \epsilon^2 P(\xi,\eta,\tau),\\
\epsilon^{\frac{1}{2}}(x+t) =&\eta + \epsilon q(\tau) + \epsilon^2 Q(\xi,\eta,\tau),\\
\epsilon^{3/2}t =&\tau,
\end{aligned}
\tag{5.4}
$$

We assume that the field quantities can be expanded into asymptotic series in ϵ as

$$
\begin{aligned}
&S = \epsilon S_1 + \epsilon^2 S_2 + \epsilon^3 S_3 + ...,\\
&u = \epsilon u_1 + \epsilon^2 u_2 + \epsilon^3 u_3 + ...,\\
&p(\tau) = p_0(\tau) + \epsilon p_1(\tau) + \epsilon^2 p_2(\tau) + \epsilon^3 p_3(\tau) + ...\ ,\\
&q(\tau) = q_0(\tau) + \epsilon q_1(\tau) + \epsilon^2 q_2(\tau) + \epsilon^3 q_3(\tau) + ...\ ,\\
&P(\xi,\eta,\tau) = P_0(\xi,\eta,\tau) + \epsilon P_1(\xi,\eta,\tau) + ...\ ,\\
&Q(\xi,\eta,\tau) = Q_0(\xi,\eta,\tau) + \epsilon Q_1(\xi,\eta,\tau) + ...\ .
\end{aligned}
\tag{5.5}
$$

Introducing (4.4) and (5.5) into equations (5.1)-(5.3) and setting the coefficients of like powers of ϵ equal to zero the following sets of equations are obtained:

$\mathcal{O}(\epsilon)$ *equations:*

$$
\begin{aligned}
&\frac{\partial S_1}{\partial \eta} - \frac{\partial S_1}{\partial \xi} + \frac{\partial u_1}{\partial \eta} + \frac{\partial u_1}{\partial \xi} = 0,\\
&\frac{\partial \pi_1}{\partial \eta} + \frac{\partial \pi_1}{\partial \xi} + \frac{\partial u_1}{\partial \eta} - \frac{\partial u_1}{\partial \xi} = 0, \quad \pi_1 = S_1,
\end{aligned}
\tag{5.6}
$$

$\mathcal{O}\left(\epsilon^2\right)$ *equations:*

$$\begin{aligned}
&\frac{\partial S_2}{\partial \eta} - \frac{\partial S_2}{\partial \xi} + \frac{\partial u_2}{\partial \eta} + \frac{\partial u_2}{\partial \xi} + \frac{\partial S_1}{\partial \tau} + \frac{\partial}{\partial \eta}(S_1 u_1) + \frac{\partial}{\partial \xi}(S_1 u_1) = 0,\\
&\frac{\partial \pi_2}{\partial \eta} + \frac{\partial \pi_2}{\partial \xi} + \frac{\partial u_2}{\partial \eta} - \frac{\partial u_2}{\partial \xi} + \frac{\partial u_1}{\partial \tau} + \frac{1}{2}\frac{\partial}{\partial \eta}(u_1^2) + \frac{1}{2}\frac{\partial}{\partial \xi}(u_1^2) = 0,\\
&\pi_2 = S_2 - 2\frac{\partial^2 S_1}{\partial \xi \partial \eta} + \frac{\partial^2 S_1}{\partial \eta^2} + \frac{\partial^2 S_1}{\partial \xi^2} + \left(\frac{\alpha - 2}{2}\right) S_1^2,
\end{aligned} \tag{5.7}$$

$\mathcal{O}\left(\epsilon^3\right)$ *equations:*

$$\begin{aligned}
&\frac{\partial S_3}{\partial \eta} - \frac{\partial S_3}{\partial \xi} + \frac{\partial u_3}{\partial \eta} + \frac{\partial u_3}{\partial \xi} + \frac{\partial S_2}{\partial \tau} + \frac{\partial}{\partial \eta}(S_1 u_2) + \frac{\partial}{\partial \xi}(S_1 u_2) + \frac{\partial}{\partial \eta}(u_1 S_2)\\
&+ \frac{\partial}{\partial \xi}(u_1 S_2) - \frac{dp_0}{d\tau}\frac{\partial S_1}{\partial \xi} - \frac{dq_0}{d\tau}\frac{\partial S_1}{\partial \eta} + \frac{\partial P_0}{\partial \xi}\frac{\partial}{\partial \eta}(u_1 + S_1)\\
&- \frac{\partial P_0}{\partial \eta}\frac{\partial}{\partial \xi}(u_1 + S_1) - \frac{\partial Q_0}{\partial \xi}\frac{\partial}{\partial \eta}(u_1 - S_1) + \frac{\partial Q_0}{\partial \eta}\frac{\partial}{\partial \xi}(u_1 - S_1) = 0,
\end{aligned} \tag{5.8}$$

$$\begin{aligned}
&\frac{\partial \pi_3}{\partial \eta} + \frac{\partial \pi_3}{\partial \xi} + \frac{\partial u_3}{\partial \eta} - \frac{\partial u_3}{\partial \xi} + \frac{\partial u_2}{\partial \tau} + \frac{\partial}{\partial \eta}(u_1 u_2) + \frac{\partial}{\partial \xi}(u_1 u_2)\\
&- \frac{dp_0}{d\tau}\frac{\partial u_1}{\partial \xi} - \frac{dq_0}{d\tau}\frac{\partial u_1}{\partial \eta} + \frac{\partial P_0}{\partial \xi}\frac{\partial}{\partial \eta}(u_1 + \pi_1) - \frac{\partial P_0}{\partial \eta}\frac{\partial}{\partial \xi}(u_1 + \pi_1)\\
&+ \frac{\partial Q_0}{\partial \xi}\frac{\partial}{\partial \eta}(u_1 - \pi_1) - \frac{\partial Q_0}{\partial \eta}\frac{\partial}{\partial \xi}(u_1 - \pi_1) = 0,
\end{aligned} \tag{5.9}$$

$$\begin{aligned}
\pi_3 =& S_3 + (\alpha - 2) S_1 S_2 + \left(\frac{3 - 2\alpha}{4}\right) S_1^3 - 2\frac{\partial^2 S_2}{\partial \xi \partial \eta} + \frac{\partial^2 S_2}{\partial \eta^2} + \frac{\partial^2 S_2}{\partial \xi^2}\\
&- 2\frac{\partial^2 S_1}{\partial \xi \partial \tau} + 2\frac{\partial^2 S_1}{\partial \eta \partial \tau} - \frac{1}{2} S_1 \left(\frac{\partial^2 S_1}{\partial \xi^2} + \frac{\partial^2 S_1}{\partial \eta^2} - 2\frac{\partial^2 S_1}{\partial \xi \partial \eta}\right).
\end{aligned} \tag{5.10}$$

$\mathcal{O}(\epsilon^4)$ equations can be found in Appendix D.

5.4 Solution of the field equations

From the solution of the equation set (5.6) we obtain

$$
\begin{aligned}
u_1 &= f_1(\xi,\tau) + g_1(\eta,\tau), \\
S_1 = \pi_1 &= f_1(\xi,\tau) - g_1(\eta,\tau),
\end{aligned} \tag{5.11}
$$

where $f_1(\xi,\tau)$ and $g_1(\eta,\tau)$ are two unknown functions whose governing equations will be obtained from the higher order perturbation expansion. Introducing (5.11) into (5.7) and then adding and subtracting the resulting equations side by side we obtain

$$
\begin{aligned}
&2\frac{\partial}{\partial\eta}(u_2+S_2) + \left[2\frac{\partial f_1}{\partial\tau} + (\alpha+1)f_1\frac{\partial f_1}{\partial\xi} + \frac{\partial^3 f_1}{\partial\xi^3}\right] + (\alpha-3)g_1\frac{\partial g_1}{\partial\eta} \\
&-(\alpha-3)f_1\frac{\partial g_1}{\partial\eta} - (\alpha-3)\frac{\partial f_1}{\partial\xi}g_1 - \frac{\partial^3 g_1}{\partial\eta^3} = 0,
\end{aligned} \tag{5.12}
$$

$$
\begin{aligned}
&2\frac{\partial}{\partial\xi}(u_2-S_2) - \left[2\frac{\partial g_1}{\partial\tau} + (\alpha+1)g_1\frac{\partial g_1}{\partial\eta} - \frac{\partial^3 g_1}{\partial\eta^3}\right] - (\alpha-3)f_1\frac{\partial f_1}{\partial\xi} \\
&+(\alpha-3)g_1\frac{\partial f_1}{\partial\xi} + (\alpha-3)f_1\frac{\partial g_1}{\partial\eta} - \frac{\partial^3 f_1}{\partial\xi^3} = 0.
\end{aligned} \tag{5.13}
$$

Integrating the equation (5.12) with respect to η and (5.13) with respect to ξ we have

$$
\begin{aligned}
(u_2+S_2) =& -\eta\left[\frac{\partial f_1}{\partial\tau} + \left(\frac{\alpha+1}{2}\right)f_1\frac{\partial f_1}{\partial\xi} + \frac{1}{2}\frac{\partial^3 f_1}{\partial\xi^3}\right] + \frac{1}{2}\frac{\partial^2 g_1}{\partial\eta^2} \\
&+\left(\frac{\alpha-3}{2}\right)\left[M(\eta,\tau)\frac{\partial f_1}{\partial\xi} + f_1 g_1 - \frac{g_1^2}{2}\right] + 2f_2(\xi,\tau),
\end{aligned} \tag{5.14}
$$

$$
\begin{aligned}
(u_2-S_2) =& \xi\left[\frac{\partial g_1}{\partial\tau} + \left(\frac{\alpha+1}{2}\right)g_1\frac{\partial g_1}{\partial\eta} - \frac{1}{2}\frac{\partial^3 g_1}{\partial\eta^3}\right] + \frac{1}{2}\frac{\partial^2 f_1}{\partial\xi^2} \\
&+\left(\frac{\alpha-3}{2}\right)\left[\frac{f_1^2}{2} - f_1 g_1 - N(\xi,\tau)\frac{\partial g_1}{\partial\eta}\right] + 2g_2(\eta,\tau),
\end{aligned} \tag{5.15}
$$

where $f_2(\xi,\tau)$ and $g_2(\eta,\tau)$ are new unknown functions, $M(\eta,\tau)$ and $N(\xi,\tau)$ are defined by

$$M(\eta,\tau)=\int^{\eta} g_1(\eta^{'},\tau)d\eta^{'}, \qquad N(\xi,\tau)=\int^{\xi} f_1(\xi^{'},\tau)d\xi^{'}. \tag{5.16}$$

As is seen from equations (5.14) and (5.15) the terms proportional to ξ and η cause secularity; therefore, the coefficients of them must vanish, which yields

$$\frac{\partial f_1}{\partial \tau}+\left(\frac{\alpha+1}{2}\right)f_1\frac{\partial f_1}{\partial \xi}+\frac{1}{2}\frac{\partial^3 f_1}{\partial \xi^3}=0, \tag{5.17}$$

$$\frac{\partial g_1}{\partial \tau}+\left(\frac{\alpha+1}{2}\right)g_1\frac{\partial g_1}{\partial \eta}-\frac{1}{2}\frac{\partial^3 g_1}{\partial \eta^3}=0. \tag{5.18}$$

Based on the statement by Su and Mirie [6], given in the Section 5.1, Xue [7] stated that the terms $M(\eta,\tau)\partial f_1/\partial\xi$ and $N(\xi,\tau)\partial g_1/\partial\eta$ appearing in equations (5.14) and (5.15) do not cause any secularity at this order but it will cause secularity in the next order equations; therefore, there should be some terms of order ϵ in the trajectory functions to eliminate these terms. As will be shown in the solution of the next order equations these terms do not cause any secularity. It is that reason, in the present work we assumed that the order of the trajectory function is ϵ^2 rather than ϵ.

Then from the solution of equations (5.14) and (5.15) we obtain u_2 and S_2 as

$$\begin{aligned}
u_2 =& f_2(\xi,\tau)+g_2(\eta,\tau)+\left(\frac{\alpha-3}{4}\right)\Bigg[M(\eta,\tau)\frac{\partial f_1}{\partial \xi}-N(\xi,\tau)\frac{\partial g_1}{\partial \eta}\\
&+\frac{1}{2}\left(f_1^2-g_1^2\right)\Bigg]+\frac{1}{4}\left(\frac{\partial^2 f_1}{\partial \xi^2}+\frac{\partial^2 g_1}{\partial \eta^2}\right),\\
S_2 =& f_2(\xi,\tau)-g_2(\eta,\tau)+\left(\frac{\alpha-3}{4}\right)\Bigg[M(\eta,\tau)\frac{\partial f_1}{\partial \xi}+N(\xi,\tau)\frac{\partial g_1}{\partial \eta}\\
&+2f_1g_1-\frac{1}{2}\left(f_1^2+g_1^2\right)\Bigg]-\frac{1}{4}\left(\frac{\partial^2 f_1}{\partial \xi^2}-\frac{\partial^2 g_1}{\partial \eta^2}\right).
\end{aligned} \tag{5.19}$$

The evolution equations (5.17) and (5.18) are the conventional Korteweg-de Vries equations, which are different from those of Xue [7], who employed the same set of tube-fluid equations. These evolution equations admit the solitary wave solution of the form

$$f_1 = A\,\mathrm{sech}^2\,\zeta_+, \quad \zeta_+ = \left[\frac{(\alpha+1)A}{12}\right]^{1/2}\left(\xi - \frac{(\alpha+1)}{6}A\tau\right),$$
$$g_1 = -B\,\mathrm{sech}^2\,\zeta_-, \quad \zeta_- = \left[\frac{(\alpha+1)B}{12}\right]^{1/2}\left(\eta + \frac{(\alpha+1)}{6}B\tau\right), \tag{5.20}$$

where A and B are constant amplitudes of the waves.

For the type of solutions given in (5.20) the functions $M(\eta,\tau)$ and $N(\xi,\tau)$ will be of the form $\tanh\zeta_\pm$. The integral of them leads to secularities as $\xi(\eta)\to\pm\infty$.

Substituting (5.11) and (5.19) into the set of equations (5.8)-(5.10), then adding and subtracting equations (5.8) and (5.9), we obtain

$$\begin{aligned}
&2\frac{\partial}{\partial\eta}(u_3+S_3) + 2\frac{\partial f_2}{\partial\tau} + (\alpha+1)\frac{\partial}{\partial\xi}(f_1f_2) + \frac{\partial^3 f_2}{\partial\xi^3} + \frac{(\alpha+4)}{2}f_1\frac{\partial^3 f_1}{\partial\xi^3}\\
&+\frac{(4\alpha+11)}{2}\frac{\partial f_1}{\partial\xi}\frac{\partial^2 f_1}{\partial\xi^2} - \frac{3}{8}(\alpha^2-2\alpha+3)f_1^2\frac{\partial f_1}{\partial\xi} + \frac{3}{4}\frac{\partial^5 f_1}{\partial\xi^5}\\
&-2\frac{dp_0}{d\tau}\frac{\partial f_1}{\partial\xi} - \frac{(\alpha-3)^2}{4}\left(\frac{\partial f_1}{\partial\xi}\frac{\partial g_1}{\partial\eta} + \frac{\partial f_1^2}{\partial\xi^2}g_1\right)M - (\alpha-3)\left(\frac{\partial f_1}{\partial\xi}g_2\right.\\
&\left.+f_1\frac{\partial g_2}{\partial\eta} - \frac{\partial}{\partial\eta}(g_1g_2) + f_2\frac{\partial g_1}{\partial\eta}\right) - \frac{\partial^3 g_2}{\partial\eta^3} - \frac{(4\alpha^2-14\alpha+15)}{4}\frac{\partial f_1}{\partial\xi}g_1^2\\
&-\left((\alpha-3)\frac{\partial f_2}{\partial\xi} - \frac{(5\alpha^2-10\alpha+3)}{4}f_1\frac{\partial f_1}{\partial\xi} - \frac{(\alpha-1)}{4}\frac{\partial^3 f_1}{\partial\xi^3}\right)g_1\\
&-\frac{(\alpha-3)}{4}\left((\alpha-3)\frac{\partial}{\partial\eta}\left(g_1\frac{\partial g_1}{\partial\eta}\right) - (\alpha-3)\frac{\partial f_1}{\partial\xi}\frac{\partial g_1}{\partial\eta} - \frac{\partial^4 g_1}{\partial\eta^4}\right.\\
&\left.-(\alpha-3)f_1\frac{\partial^2 g_1}{\partial\eta^2}\right)N + \frac{(5\alpha^2-10\alpha+3)}{8}g_1^2\frac{\partial g_1}{\partial\eta} + \frac{(3\alpha-3)}{4}f_1\frac{\partial^3 g_1}{\partial\eta^3}\\
&-\frac{(7\alpha^2-22\alpha+21)}{8}\left(2f_1g_1\frac{\partial g_1}{\partial\eta} - f_1^2\frac{\partial g_1}{\partial\eta}\right) - \frac{(\alpha-5)}{4}\frac{\partial^2 f_1}{\partial\xi^2}\frac{\partial g_1}{\partial\eta}\\
&+\frac{3}{2}\frac{\partial f_1}{\partial\xi}\frac{\partial^2 g_1}{\partial\eta^2} + \frac{5}{2}\frac{\partial}{\partial\eta}\left(g_1\frac{\partial^2 g_1}{\partial\eta^2}\right) + \frac{(5\alpha+9)}{8}\frac{\partial}{\partial\eta}\left[\left(\frac{\partial g_1}{\partial\eta}\right)^2\right]\\
&-\frac{1}{2}\frac{\partial^5 g_1}{\partial\eta^5} - 4\frac{\partial P_0}{\partial\eta}\frac{\partial f_1}{\partial\xi} = 0,
\end{aligned} \tag{5.21}$$

$$
\begin{aligned}
&2\frac{\partial}{\partial \xi}(u_3 - S_3) - 2\frac{\partial g_2}{\partial \tau} - (\alpha+1)\frac{\partial}{\partial \eta}(g_1 g_2) + \frac{\partial^3 g_2}{\partial \eta^3} - \frac{(\alpha+4)}{2} g_1 \frac{\partial^3 g_1}{\partial \eta^3} \\
&- \frac{(4\alpha+11)}{2}\frac{\partial g_1}{\partial \eta}\frac{\partial^2 g_1}{\partial \eta^2} - \frac{3}{8}(\alpha^2 - 2\alpha + 3) g_1^2 \frac{\partial g_1}{\partial \eta} + \frac{3}{4}\frac{\partial^5 g_1}{\partial \eta^5} \\
&+ 2\frac{dq_0}{d\tau}\frac{\partial g_1}{\partial \eta} - \frac{(\alpha-3)^2}{4}\left(\frac{\partial f_1}{\partial \xi}\frac{\partial g_1}{\partial \eta} + \frac{\partial g_1^2}{\partial \eta^2} f_1\right) N + (\alpha - 3)\left(\frac{\partial g_1}{\partial \eta} f_2\right. \\
&\left. + g_1 \frac{\partial f_2}{\partial \xi} - \frac{\partial}{\partial \xi}(f_1 f_2) + g_2 \frac{\partial f_1}{\partial \xi}\right) - \frac{\partial^3 f_2}{\partial \xi^3} - \frac{(4\alpha^2 - 14\alpha + 15)}{4}\frac{\partial g_1}{\partial \eta} f_1^2 \\
&+ \left((\alpha-3)\frac{\partial g_2}{\partial \eta} + \frac{(5\alpha^2 - 10\alpha + 3)}{4} g_1 \frac{\partial g_1}{\partial \eta} - \frac{(\alpha-1)}{4}\frac{\partial^3 g_1}{\partial \eta^3}\right) f_1 \\
&- \frac{(\alpha-3)}{4}\left((\alpha-3)\frac{\partial}{\partial \xi}\left(f_1 \frac{\partial f_1}{\partial \xi}\right) - (\alpha-3)\frac{\partial f_1}{\partial \xi}\frac{\partial g_1}{\partial \eta} + \frac{\partial^4 f_1}{\partial \xi^4}\right. \\
&\left. - (\alpha-3) g_1 \frac{\partial^2 f_1}{\partial \xi^2}\right) M + \frac{(5\alpha^2 - 10\alpha + 3)}{8} f_1^2 \frac{\partial f_1}{\partial \xi} - \frac{(3\alpha-3)}{4} g_1 \frac{\partial^3 f_1}{\partial \xi^3} \\
&- \frac{(7\alpha^2 - 22\alpha + 21)}{8}\left(2 g_1 f_1 \frac{\partial f_1}{\partial \xi} - g_1^2 \frac{\partial f_1}{\partial \xi}\right) + \frac{(\alpha-5)}{4}\frac{\partial^2 g_1}{\partial \eta^2}\frac{\partial f_1}{\partial \xi} \\
&- \frac{3}{2}\frac{\partial g_1}{\partial \eta}\frac{\partial^2 f_1}{\partial \xi^2} - \frac{5}{2}\frac{\partial}{\partial \xi}\left(f_1 \frac{\partial^2 f_1}{\partial \xi^2}\right) - \frac{(5\alpha+9)}{8}\frac{\partial}{\partial \xi}\left[\left(\frac{\partial f_1}{\partial \xi}\right)^2\right] \\
&- \frac{1}{2}\frac{\partial^5 f_1}{\partial \xi^5} - 4\frac{\partial Q_0}{\partial \xi}\frac{\partial g_1}{\partial \eta} = 0. \qquad (5.22)
\end{aligned}
$$

Integrating (5.21) with respect to η and (5.22) with respect to ξ we obtain

$$
\begin{aligned}
&2(u_3 + S_3) + \eta\left(2\frac{\partial f_2}{\partial \tau} + (\alpha+1)\frac{\partial}{\partial \xi}(f_1 f_2) + \frac{\partial^3 f_2}{\partial \xi^3} + \frac{(\alpha+4)}{2} f_1 \frac{\partial^3 f_1}{\partial \xi^3}\right. \\
&+ \frac{(4\alpha+11)}{2}\frac{\partial f_1}{\partial \xi}\frac{\partial^2 f_1}{\partial \xi^2} - \frac{3}{8}(\alpha^2 - 2\alpha + 3) f_1^2 \frac{\partial f_1}{\partial \xi} + \frac{3}{4}\frac{\partial^5 f_1}{\partial \xi^5} \\
&\left. - 2\frac{dp_0}{d\tau}\frac{\partial f_1}{\partial \xi}\right) - \frac{(\alpha-3)^2}{4}\left(\frac{\partial f_1}{\partial \xi}\int^{\eta}\left(\frac{\partial g_1}{\partial \eta} M\right) d\eta' + \frac{\partial f_1^2}{\partial \xi^2}\int^{\eta}(g_1 M) d\eta'\right) \\
&- (\alpha-3)\frac{\partial f_1}{\partial \xi}\int^{\eta} g_2 d\eta' - (\alpha-3)(f_1 g_2 - g_1 g_2 + f_2 g_1) - \frac{\partial^2 g_2}{\partial \eta^2} \\
&- \frac{(4\alpha^2 - 14\alpha + 15)}{4}\frac{\partial f_1}{\partial \xi}\int^{\eta} g_1^2 d\eta' - \left((\alpha-3)\frac{\partial f_2}{\partial \xi} - \frac{(\alpha-1)}{4}\frac{\partial^3 f_1}{\partial \xi^3}\right. \\
&\left. - \frac{(5\alpha^2 - 10\alpha + 3)}{4} f_1 \frac{\partial f_1}{\partial \xi}\right) M - \frac{(\alpha-3)}{4}\left((\alpha-3) g_1 \frac{\partial g_1}{\partial \eta} - \frac{\partial^3 g_1}{\partial \eta^3}\right. \\
&\left. - (\alpha-3)\frac{\partial f_1}{\partial \xi} g_1 - (\alpha-3) f_1 \frac{\partial g_1}{\partial \eta}\right) N + \frac{(5\alpha^2 - 10\alpha + 3)}{24} g_1^3
\end{aligned}
$$

$$
+\frac{(3\alpha-3)}{4}f_1\frac{\partial^2 g_1}{\partial\eta^2}-\frac{(7\alpha^2-22\alpha+21)}{8}\left(f_1g_1^2-f_1^2g_1\right)
$$
$$
-\frac{(\alpha-5)}{4}\frac{\partial^2 f_1}{\partial\xi^2}g_1+\frac{3}{2}\frac{\partial f_1}{\partial\xi}\frac{\partial g_1}{\partial\eta}+\frac{5}{2}g_1\frac{\partial^2 g_1}{\partial\eta^2}+\frac{(5\alpha+9)}{8}\left(\frac{\partial g_1}{\partial\eta}\right)^2
$$
$$
-\frac{1}{2}\frac{\partial^4 g_1}{\partial\eta^5}-4P_0\frac{\partial f_1}{\partial\xi}=2f_3(\xi,\tau), \tag{5.23}
$$

$$
2(u_3-S_3)+\xi\left(-2\frac{\partial g_2}{\partial\tau}-(\alpha+1)\frac{\partial}{\partial\eta}(g_1g_2)+\frac{\partial^3 g_2}{\partial\eta^3}-\frac{(\alpha+4)}{2}g_1\frac{\partial^3 g_1}{\partial\eta^3}\right.
$$
$$
-\frac{(4\alpha+11)}{2}\frac{\partial g_1}{\partial\eta}\frac{\partial^2 g_1}{\partial\eta^2}-\frac{3}{8}(\alpha^2-2\alpha+3)g_1^2\frac{\partial g_1}{\partial\eta}+\frac{3}{4}\frac{\partial^5 g_1}{\partial\eta^5}
$$
$$
\left.+2\frac{dq_0}{d\tau}\frac{\partial g_1}{\partial\eta}\right)-\frac{(\alpha-3)^2}{4}\left(\frac{\partial g_1}{\partial\eta}\int^{\xi}\left(\frac{\partial f_1}{\partial\xi}N\right)d\xi'+\frac{\partial g_1^2}{\partial\eta^2}\int^{\xi}(f_1N)\,d\xi'\right)
$$
$$
+(\alpha-3)\frac{\partial g_1}{\partial\eta}\int^{\xi}f_2d\xi'+(\alpha-3)\left(g_1f_2-f_1f_2+g_2f_1\right)-\frac{\partial^2 f_2}{\partial\xi^2}
$$
$$
-\frac{(4\alpha^2-14\alpha+15)}{4}\frac{\partial g_1}{\partial\eta}\int^{\xi}f_1^2d\xi'+\left((\alpha-3)\frac{\partial g_2}{\partial\eta}-\frac{(\alpha-1)}{4}\frac{\partial^3 g_1}{\partial\eta^3}\right.
$$
$$
\left.+\frac{(5\alpha^2-10\alpha+3)}{4}g_1\frac{\partial g_1}{\partial\eta}\right)N-\frac{(\alpha-3)}{4}\left((\alpha-3)f_1\frac{\partial f_1}{\partial\xi}+\frac{\partial^3 f_1}{\partial\xi^3}\right.
$$
$$
\left.-(\alpha-3)f_1\frac{\partial g_1}{\partial\eta}-(\alpha-3)\frac{\partial f_1}{\partial\xi}g_1\right)M+\frac{(5\alpha^2-10\alpha+3)}{24}f_1^3
$$
$$
-\frac{(3\alpha-3)}{4}g_1\frac{\partial^2 f_1}{\partial\xi^2}-\frac{(7\alpha^2-22\alpha+21)}{8}\left(f_1^2g_1-f_1g_1^2\right)
$$
$$
+\frac{(\alpha-5)}{4}\frac{\partial^2 g_1}{\partial\eta^2}f_1-\frac{3}{2}\frac{\partial g_1}{\partial\eta}\frac{\partial f_1}{\partial\xi}-\frac{5}{2}f_1\frac{\partial^2 f_1}{\partial\xi^2}-\frac{(5\alpha+9)}{8}\left(\frac{\partial f_1}{\partial\xi}\right)^2
$$
$$
-\frac{1}{2}\frac{\partial^4 f_1}{\partial\xi^4}-4Q_0\frac{\partial g_1}{\partial\eta}=2g_3(\eta,\tau), \tag{5.24}
$$

where $f_3(\xi,\tau)$ and $g_3(\eta,\tau)$ are two unknown functions whose evolution equations will be obtained from next order equations. In order to remove the secularity caused by the terms proportional to ξ and η, the coefficient of η in (5.23) and the coefficient of ξ in (5.24) must vanish, which yields

$$
\frac{\partial f_2}{\partial\tau}+\frac{(\alpha+1)}{2}\frac{\partial}{\partial\xi}(f_1f_2)+\frac{1}{2}\frac{\partial^3 f_2}{\partial\xi^3}=R_1(\xi,\tau), \tag{5.25}
$$

$$\frac{\partial g_2}{\partial \tau} + \frac{(\alpha+1)}{2}\frac{\partial}{\partial \eta}(g_1 g_2) - \frac{1}{2}\frac{\partial^3 g_2}{\partial \eta^3} = T_1(\eta,\tau), \tag{5.26}$$

where $R_1(\xi,\tau)$ and $T_1(\eta,\tau)$ are given as follows

$$\begin{aligned} R_1(\xi,\tau) =& \frac{3}{16}(\alpha^2-2\alpha+3)f_1^2\frac{\partial f_1}{\partial \xi} - \frac{(4\alpha+11)}{4}\frac{\partial f_1}{\partial \xi}\frac{\partial^2 f_1}{\partial \xi^2} \\ &- \frac{(\alpha+4)}{4}f_1\frac{\partial^3 f_1}{\partial \xi^3} - \frac{3}{8}\frac{\partial^5 f_1}{\partial \xi^5} + \frac{dp_0}{d\tau}\frac{\partial f_1}{\partial \xi}, \end{aligned} \tag{5.27}$$

$$\begin{aligned} T_1(\eta,\tau) =& -\frac{3}{16}(\alpha^2-2\alpha+3)g_1^2\frac{\partial g_1}{\partial \eta} - \frac{(4\alpha+11)}{4}\frac{\partial g_1}{\partial \eta}\frac{\partial^2 g_1}{\partial \eta^2} \\ &- \frac{(\alpha+4)}{4}g_1\frac{\partial^3 g_1}{\partial \eta^3} + \frac{3}{8}\frac{\partial^5 g_1}{\partial \eta^5} + \frac{dq_0}{d\tau}\frac{\partial g_1}{\partial \eta}. \end{aligned} \tag{5.28}$$

By using the results given in (5.16) and (5.20) the following identities can be obtained for the terms involving f_1, N, g_1 and M,

$$\begin{aligned} \frac{\partial^2 f_1}{\partial \xi^2} &= \frac{(\alpha+1)}{6}(2Af_1 - 3f_1^2), \quad \left(\frac{\partial f_1}{\partial \xi}\right)^2 = \frac{(\alpha+1)}{3}(Af_1^2 - f_1^3), \\ \frac{\partial^4 f_1}{\partial \xi^4} &= \frac{(\alpha+1)^2}{18}(2A^2 f_1 - 15Af_1^2 + 15f_1^3), \\ \frac{\partial^6 f_1}{\partial \xi^6} &= (\alpha+1)^3\left(\frac{A^3}{27}f_1 - \frac{7A^2}{6}f_1^2 + \frac{35A}{9}f_1^3 - \frac{35}{12}f_1^4\right), \\ \int f_1^2 d\xi' &= \frac{N}{3}\left(f_1 + 2A\right), \qquad \int f_1 N d\xi' = -\left(\frac{6}{\alpha+1}\right) f_1, \\ \int \left(\frac{\partial f_1}{\partial \xi}N\right) d\xi' &= \frac{2N}{3}(f_1 - A), \end{aligned} \tag{5.29}$$

$$\begin{aligned} \frac{\partial^2 g_1}{\partial \eta^2} &= \frac{(\alpha+1)}{6}(2Bg_1 + 3g_1^2), \quad \left(\frac{\partial g_1}{\partial \eta}\right)^2 = \frac{(\alpha+1)}{3}(Bg_1^2 + g_1^3), \\ \frac{\partial^4 g_1}{\partial \eta^4} &= \frac{(\alpha+1)^2}{18}(2B^2 g_1 + 15Bg_1^2 + 15g_1^3), \\ \frac{\partial^6 g_1}{\partial \eta^6} &= (\alpha+1)^3\left(\frac{B^3}{27}g_1 + \frac{7B^2}{6}g_1^2 + \frac{35B}{9}g_1^3 + \frac{35}{12}g_1^4\right), \\ \int g_1^2 d\eta' &= \frac{M}{3}\left(g_1 - 2B\right), \qquad \int g_1 M d\eta' = \left(\frac{6}{\alpha+1}\right) g_1, \\ \int \left(\frac{\partial g_1}{\partial \eta}M\right) d\eta' &= \frac{2M}{3}(g_1 + B). \end{aligned} \tag{5.30}$$

Then the equations (5.25) and (5.26) can be written as

$$\frac{\partial f_2}{\partial \tau} + \frac{(\alpha+1)}{2}\frac{\partial}{\partial \xi}(f_1 f_2) + \frac{1}{2}\frac{\partial^3 f_2}{\partial \xi^3} = \frac{\partial}{\partial \xi}(R_2(f_1)), \tag{5.31}$$

$$\frac{\partial g_2}{\partial \tau} + \frac{(\alpha+1)}{2}\frac{\partial}{\partial \eta}(g_1 g_2) - \frac{1}{2}\frac{\partial^3 g_2}{\partial \eta^3} = \frac{\partial}{\partial \eta}(T_2(g_1)), \tag{5.32}$$

where $R_2(f_1)$ and $T_2(g_1)$ are defined as follows

$$R_2(f_1) = \left(\frac{dp_0}{d\tau} - \frac{(\alpha+1)^2}{24}A^2\right) f_1 + \frac{5}{48}(\alpha+1)(\alpha-3)Af_1^2 + \left(\frac{7\alpha+16}{24}\right) f_1^3, \tag{5.33}$$

$$T_2(g_1) = \left(\frac{dq_0}{d\tau} + \frac{(\alpha+1)^2}{24}B^2\right) g_1 + \frac{5}{48}(\alpha+1)(\alpha-3)Bg_1^2 - \left(\frac{7\alpha+16}{24}\right) g_1^3. \tag{5.34}$$

As is seen from the equations (5.23) and (5.24) the other terms in the expression of u_3 and S_3 do not cause any secularity of the type $\int M(\eta')d\eta'$ and $\int N(\xi')d\xi'$ for this order, but it might have secularities in the next order.

Seeking a progressive wave solution for the equations (5.31) and (5.32) of the form $f_2 = f_2(\zeta_+)$, $g_2 = g_2(\zeta_-)$, we have

$$\frac{(\alpha+1)}{24}Af_2''' + \frac{(\alpha+1)}{2}(f_1 f_2)' - \frac{(\alpha+1)}{6}Af_2' = R_2'(f_1), \tag{5.35}$$

$$-\frac{(\alpha+1)}{24}Bg_2''' + \frac{(\alpha+1)}{2}(g_1 g_2)' + \frac{(\alpha+1)}{6}Bg_2' = T_2'(g_1). \tag{5.36}$$

Using localization condition and integrating once with respect to ζ_+ and ζ_-, respectively, one obtains

$$\frac{(\alpha+1)}{24}Af_2'' + \frac{(\alpha+1)}{6}(3f_1 - A)f_2 = R_2(f_1), \tag{5.37}$$

$$-\frac{(\alpha+1)}{24}Bg_2'' + \frac{(\alpha+1)}{6}(3g_1 + B)g_2 = T_2(g_1). \tag{5.38}$$

The first terms in the expression of $R_2(f_1)$ and $T_2(g_1)$ cause to secularity;

therefore the coefficients of f_1 and g_1 must vanish, which yields

$$p_0 = \frac{(\alpha+1)^2}{24} A^2 \tau, \qquad q_0 = -\frac{(\alpha+1)^2}{24} B^2 \tau. \tag{5.39}$$

Here $A^2(\alpha+1)^2/24$ and $-B^2(\alpha+1)^2/24$ correspond to the speed correction terms for the right and left going waves, respectively. We shall propose a solution for f_2 and g_2 of the following form

$$f_2 = a_1 \operatorname{sech}^4 \zeta_+ + a_2 \operatorname{sech}^2 \zeta_+, \tag{5.40}$$

$$g_2 = b_1 \operatorname{sech}^4 \zeta_- + b_2 \operatorname{sech}^2 \zeta_-, \tag{5.41}$$

where a_i and b_i are constants to be determined from the solutions of (5.37) and (5.38), respectively. Evaluating the derivative of F_1 and G_1 we have

$$f_2^{''} = -20a_1 \operatorname{sech}^6 \zeta_+ + (16a_1 - 6a_2) \operatorname{sech}^4 \zeta_+ + 4a_2 \operatorname{sech}^2 \zeta_+, \tag{5.42}$$

$$g_2^{''} = -20b_1 \operatorname{sech}^6 \zeta_- + (16b_1 - 6b_2) \operatorname{sech}^4 \zeta_- + 4b_2 \operatorname{sech}^2 \zeta_-. \tag{5.43}$$

Inserting (5.40) and (5.42) into (5.37) and inserting (5.41) and (5.43) into (5.38) and setting the coefficients of $\operatorname{sech}^6 \zeta_+$ and $\operatorname{sech}^4 \zeta_+$ equal to zero, one has

$$a_1 = -\frac{(7\alpha+16)}{8(\alpha+1)} A^2, \quad a_2 = \frac{(5\alpha^2+11\alpha+33)}{12(\alpha+1)} A^2, \tag{5.44}$$

$$b_1 = \frac{(7\alpha+16)}{8(\alpha+1)} B^2, \quad b_2 = -\frac{(5\alpha^2+11\alpha+33)}{12(\alpha+1)} B^2, \tag{5.45}$$

Then the particular solution of the differential equations (5.37) and (5.38) may be given by

$$\begin{aligned} f_2 &= \frac{1}{24(\alpha+1)} \left[(5\alpha^2+11\alpha+33)2Af_1 - (7\alpha+16)3f_1^2\right], \\ g_2 &= \frac{1}{24(\alpha+1)} \left[(5\alpha^2+11\alpha+33)2Bg_1 + (7\alpha+16)3g_1^2\right]. \end{aligned} \tag{5.46}$$

Then the equations (5.23) and (5.24) can be written in the following form

$$
\begin{aligned}
u_3 + S_3 =& \frac{(7\alpha+16)}{4(\alpha+1)} g_1^3 + \frac{(43\alpha^2+103\alpha+276)}{48(\alpha+1)} B g_1^2 \\
&+ \frac{(7\alpha^2+15\alpha+35)}{72} B^2 g_1 + \frac{(12\alpha^3-37\alpha^2+9\alpha-50)}{24(\alpha+1)} \\
&\times A f_1 g_1 + \frac{(2\alpha^3-7\alpha^2+3\alpha-96)}{24(\alpha+1)} B f_1 g_1 \\
&+ \frac{(4\alpha^3-11\alpha^2-3\alpha-24)}{16(\alpha+1)} f_1 g_1^2 - \frac{3}{4}\frac{\partial f_1}{\partial \xi}\frac{\partial g_1}{\partial \eta} + 2P_0 \frac{\partial f_1}{\partial \xi} \\
&- \frac{(14\alpha^3-41\alpha^2+3\alpha+22)}{16(\alpha+1)} f_1^2 g_1 \\
&+ \left[\frac{(4\alpha^3-5\alpha^2+\alpha-98)}{24(\alpha+1)} A \frac{\partial f_1}{\partial \xi} - \frac{(\alpha^3+\alpha^2-9\alpha+63)}{24(\alpha+1)} \right. \\
&\times B \frac{\partial f_1}{\partial \xi} - \frac{(4\alpha^3+\alpha^2-11\alpha-44)}{8(\alpha+1)} f_1 \frac{\partial f_1}{\partial \xi} \\
&\left. + \frac{(4\alpha^3-11\alpha^2+3\alpha+6)}{16(\alpha+1)} g_1 \frac{\partial f_1}{\partial \xi} \right] M - \frac{(\alpha-3)}{2} \left[\frac{(\alpha+1)}{12} \right. \\
&\left. \times B \frac{\partial g_1}{\partial \eta} + \frac{(\alpha-3)}{4} \left(f_1 \frac{\partial g_1}{\partial \eta} + g_1 \frac{\partial f_1}{\partial \xi} \right) + g_1 \frac{\partial g_1}{\partial \eta} \right] N \\
&+ 2 f_3(\xi, \tau),
\end{aligned} \tag{5.47}
$$

$$
\begin{aligned}
u_3 - S_3 =& \frac{(7\alpha+16)}{4(\alpha+1)} f_1^3 - \frac{(43\alpha^2+103\alpha+276)}{48(\alpha+1)} A f_1^2 \\
&+ \frac{(7\alpha^2+15\alpha+35)}{72} A^2 f_1 - \frac{(12\alpha^3-37\alpha^2+9\alpha-50)}{24(\alpha+1)} \\
&\times B f_1 g_1 - \frac{(2\alpha^3-7\alpha^2+3\alpha-96)}{24(\alpha+1)} A f_1 g_1 \\
&+ \frac{(4\alpha^3-11\alpha^2-3\alpha-24)}{16(\alpha+1)} f_1^2 g_1 + \frac{3}{4}\frac{\partial f_1}{\partial \xi}\frac{\partial g_1}{\partial \eta} + 2Q_0 \frac{\partial g_1}{\partial \eta} \\
&- \frac{(14\alpha^3-41\alpha^2+3\alpha+22)}{16(\alpha+1)} f_1 g_1^2 \\
&+ \left[- \frac{(4\alpha^3-5\alpha^2+\alpha-98)}{24(\alpha+1)} B \frac{\partial g_1}{\partial \eta} + \frac{(\alpha^3+\alpha^2-9\alpha+63)}{24(\alpha+1)} \right. \\
&\times A \frac{\partial g_1}{\partial \eta} - \frac{(4\alpha^3+\alpha^2-11\alpha-44)}{8(\alpha+1)} g_1 \frac{\partial g_1}{\partial \eta} \\
&\left. + \frac{(4\alpha^3-11\alpha^2+3\alpha+6)}{16(\alpha+1)} f_1 \frac{\partial g_1}{\partial \eta} \right] N + \frac{(\alpha-3)}{2} \left[\frac{(\alpha+1)}{12} \right.
\end{aligned}
$$

$$\times A\frac{\partial f_1}{\partial \xi} - \frac{(\alpha-3)}{4}\left(f_1\frac{\partial g_1}{\partial \eta} + g_1\frac{\partial f_1}{\partial \xi}\right) - f_1\frac{\partial f_1}{\partial \xi}\Bigg] M$$
$$+ 2g_3(\eta,\tau). \tag{5.48}$$

As might be seen from equations (5.47) and (5.48) these terms appearing in the expressions of u_2 and S_2 do not cause any secularity in the solution of u_3 and S_3. Therefore the statement by Su and Mirie [6] is incorrect. However as we stated before, some of the terms appearing in the expressions of u_3 and S_3 (the equations (5.47) and (5.48)) may cause additional secularity in the expressions of u_4 and S_4. There appears to be two types of secularity in the solution of $\mathcal{O}(\epsilon^4)$ equation. As was seen before, the first type of secularity results from the terms proportional to ξ and η which will be studied later. The second type secularity occurs from the terms proportional $\int^{\xi} N(\xi^{'},\tau)d\xi^{'}$ and $\int^{\eta} M(\eta^{'},\tau)d\eta^{'}$ as $\xi(\eta)\to\pm\infty$. Here we shall first consider only the parts of $\mathcal{O}(\epsilon^4)$ equations leading to $\int^{\eta} M(\eta^{'},\tau)d\eta^{'}$ type of secularity. Similar expressions may be valid for $\int^{\xi} N(\xi^{'},\tau)d\xi^{'}$ type of secularity.

For this purpose we consider the following part of the $\mathcal{O}(\epsilon^4)$ equation

$$\begin{aligned}
&2\frac{\partial}{\partial\eta}(u_4+S_4) + \frac{\partial}{\partial\tau}(u_3+S_3) + (\alpha-2)\frac{\partial}{\partial\xi}(S_1S_3) + \frac{\partial}{\partial\xi}(u_1S_3)\\
&+\frac{\partial}{\partial\xi}\left[u_3(u_1+S_1)\right] + \frac{\partial^3 S_3}{\partial\xi^3} + (\alpha-2)S_2\frac{\partial S_2}{\partial\xi} + u_2\frac{\partial}{\partial\xi}(u_2+S_2)\\
&+S_2\frac{\partial u_2}{\partial\xi} - \frac{dp_0}{d\tau}\frac{\partial}{\partial\xi}(u_2+S_2) - \frac{(6\alpha-9)}{4}\frac{\partial}{\partial\xi}\left(S_1^2S_2\right) - 2\frac{\partial^3 S_2}{\partial\xi^2\partial\tau}\\
&-\frac{1}{2}\frac{\partial}{\partial\xi}\left(S_1\frac{\partial^2 S_2}{\partial\xi^2} + \frac{\partial^2 S_1}{\partial\xi^2}S_2\right) + \frac{(3\alpha-4)}{2}S_1^3\frac{\partial S_1}{\partial\xi} + \frac{\partial^3 S_1}{\partial\xi\partial\tau^2}\\
&+\frac{1}{4}\frac{\partial}{\partial\xi}\left(S_1^2\frac{\partial^2 S_1}{\partial\xi^2}\right) + \frac{\partial}{\partial\xi}\left(S_1\frac{\partial^2 S_1}{\partial\xi\partial\tau}\right) + 2\frac{dp_0}{d\tau}\frac{\partial^3 S_1}{\partial\xi^3}\\
&-\frac{dp_1}{d\tau}\frac{\partial}{\partial\xi}(u_1+S_1) = 0.
\end{aligned}\tag{5.49}$$

A similar expression may be given for $2\frac{\partial}{\partial\xi}(u_4-S_4)$ equation. We split (5.49) into two parts which contain the variables u_3+S_3 and $(u_2,\ S_2,\ u_3-S_3)$, respectively. Then, we obtain:

$$
\begin{aligned}
&\frac{\partial}{\partial\tau}(u_3+S_3)+\frac{(\alpha+1)}{4}\frac{\partial}{\partial\xi}[(u_1+S_1)(u_3+S_3)]+\frac{1}{2}\frac{\partial^3}{\partial\xi^3}(u_3+S_3)\\
&=\frac{-(4\alpha^3+\alpha^2-11\alpha-44)}{8}(\alpha+1)\left[\frac{7}{8}f_1^4-\frac{7}{6}Af_1^3+\frac{1}{3}A^2f_1^2\right]M,
\end{aligned}
\tag{5.50}
$$

$$
\begin{aligned}
&(\alpha-2)S_2\frac{\partial S_2}{\partial\xi}+u_2\frac{\partial}{\partial\xi}(u_2+S_2)+S_2\frac{\partial u_2}{\partial\xi}\\
&-\frac{dp_0}{d\tau}\frac{\partial}{\partial\xi}(u_2+S_2)-\frac{(6\alpha-9)}{4}\frac{\partial}{\partial\xi}\left(S_1^2S_2\right)-2\frac{\partial^3 S_2}{\partial\xi^2\partial\tau}\\
&-\frac{1}{2}\frac{\partial}{\partial\xi}\left(S_1\frac{\partial^2 S_2}{\partial\xi^2}+\frac{\partial^2 S_1}{\partial\xi^2}S_2\right)+\frac{(3\alpha-9)}{4}\frac{\partial}{\partial\xi}\left(S_1S_3\right)\\
&-\frac{1}{2}\frac{\partial^3}{\partial\xi^3}\left(u_3-S_3\right)-\frac{(\alpha-3)}{2}\frac{\partial}{\partial\xi}\left[(u_1-S_1)S_3\right]\\
&-\frac{(\alpha-3)}{4}\frac{\partial}{\partial\xi}\left[\left(u_1+S_1\right)\left(u_3-S_3\right)\right]\\
&=\frac{(7\alpha^2-5\alpha-48)}{8}(\alpha+1)\left[\frac{7}{8}f_1^4-\frac{7}{6}Af_1^3+\frac{1}{3}A^2f_1^2\right]M.
\end{aligned}
\tag{5.51}
$$

As is seen the integration of equations (5.50) and (5.51) with respect to η leads to secularity. In order to remove secularity, we should set the coefficient of the term $f_1\frac{\partial f_1}{\partial\xi}M$ in u_3+S_3 equal to $-\frac{(7\alpha^2-5\alpha-48)}{8(\alpha+1)}$. Similar expression may be given for $\int^{\xi} N(\xi',\tau)d\xi'$ type of secularities. In order to remove these secularities the trajectory functions should have the following form:

$$
\begin{aligned}
P_0&=\frac{(2\alpha^2-5\alpha+2)}{8}f_1(\xi,\tau)M(\eta,\tau),\\
Q_0&=\frac{(2\alpha^2-5\alpha+2)}{8}g_1(\eta,\tau)N(\xi,\tau).
\end{aligned}
\tag{5.52}
$$

To obtain the secularities of type η (or ξ) we shall use the equation (5.49) to obtain the governing equation for $f_3(\xi,\tau)$. We substitute the field variables into (5.49) and integrate with respect to η, then the terms proportional to η in the resulting equation cause to secularity. In order to remove secularity, the coefficient of η must vanish, that is

$$\frac{\partial f_3}{\partial \tau} + \frac{(\alpha+1)}{2}\frac{\partial}{\partial \xi}(f_1 f_3) + \frac{1}{2}\frac{\partial^3 f_3}{\partial \xi^3} = R_3(\xi,\tau), \tag{5.53}$$

where $R_3(\xi,\tau)$ is defined as follows

$$\begin{aligned} R_3(\xi,\tau) =& \frac{\partial^3 f_2}{\partial \xi^2 \partial \tau} + \frac{3}{16}(\alpha^2 - 2\alpha + 3)\frac{\partial}{\partial \xi}\left(f_1^2 f_2\right) + \frac{(\alpha-2)}{4} f_1 \frac{\partial^3 f_2}{\partial \xi^3} \\ &+ \frac{(2\alpha-5)}{4}\frac{\partial}{\partial \xi}\left(\frac{\partial f_1}{\partial \xi}\frac{\partial f_2}{\partial \xi}\right) + \frac{(\alpha-2)}{4}\frac{\partial^3 f_1}{\partial \xi^3} f_2 \\ &- \frac{(\alpha+1)}{2} f_2 \frac{\partial f_2}{\partial \xi} + \frac{dp_0}{d\tau}\frac{\partial f_2}{\partial \xi} + \frac{1}{8}\frac{\partial^5 f_2}{\partial \xi^5} - \frac{1}{2}\frac{\partial^3 f_1}{\partial \xi \partial \tau^2} \\ &- \frac{1}{4}\frac{\partial^5 f_1}{\partial \xi^4 \partial \tau} - \frac{(\alpha-1)}{4} f_1 \frac{\partial^3 f_1}{\partial \xi^2 \partial \tau} - \frac{(\alpha-2)}{2}\frac{\partial f_1}{\partial \xi}\frac{\partial^2 f_1}{\partial \xi \partial \tau} \\ &- \frac{(\alpha-3)}{4}\frac{\partial f_1}{\partial \tau}\frac{\partial^2 f_1}{\partial \xi^2} + \frac{1}{16}\frac{\partial^7 f_1}{\partial \xi^7} + \frac{(\alpha+1)}{16} f_1 \frac{\partial^5 f_1}{\partial \xi^5} \\ &- \frac{dp_0}{d\tau}\frac{\partial^3 f_1}{\partial \xi^3} + \frac{dp_1}{d\tau}\frac{\partial f_1}{\partial \xi} + \frac{(7\alpha+31)}{32}\frac{\partial f_1}{\partial \xi}\frac{\partial^4 f_1}{\partial \xi^4} \\ &+ \frac{(7\alpha+33)}{16}\frac{\partial^2 f_1}{\partial \xi^2}\frac{\partial^3 f_1}{\partial \xi^3} + \frac{3}{8}(\alpha-3)^3 B \frac{\partial f_1}{\partial \xi}\frac{\partial^2 f_1}{\partial \xi^2} \\ &- \frac{(5\alpha^2 - 10\alpha + 21)}{64}\left(\frac{\partial f_1}{\partial \xi}\right)^3 - \frac{(3\alpha^2 - 9\alpha + 22)}{32} f_1^2 \frac{\partial^3 f_1}{\partial \xi^3} \\ &- \frac{(11\alpha^2 - 28\alpha + 65)}{32} f_1 \frac{\partial f_1}{\partial \xi}\frac{\partial^2 f_1}{\partial \xi^2} \\ &- \frac{(23\alpha^3 - 55\alpha^2 + 33\alpha - 15)}{192} f_1^3 \frac{\partial f_1}{\partial \xi}. \end{aligned} \tag{5.54}$$

Noting the equations (5.29), (5.39) and (5.46), then the equation (5.53) may be written as

$$\frac{\partial f_3}{\partial \tau} + \frac{(\alpha+1)}{2}\frac{\partial}{\partial \xi}(f_1 f_3) + \frac{1}{2}\frac{\partial^3 f_3}{\partial \xi^3} = \frac{\partial R_4(f_1)}{\partial \xi} \tag{5.55}$$

where $R_4(f_1)$ is defined as follows

$$R_4(f_1) = \left(\frac{dp_1}{d\tau} - \frac{5(\alpha+1)^3}{432}A^3\right) f_1$$

$$
\begin{aligned}
&+\left(\frac{(23\alpha^4-100\alpha^3-479\alpha^2-1700\alpha-2073)}{576(\alpha+1)}A^2\right.\\
&\left.+\frac{(\alpha+1)(\alpha-3)^2}{16}AB\right)f_1^2\\
&+\left(\frac{(134\alpha^3+1471\alpha^2+4862\alpha+5712)}{576(\alpha+1)}A\right.\\
&\left.-\frac{(\alpha+1)(\alpha-3)^2}{16}B\right)f_1^3\\
&+\frac{(-341\alpha^2-1472\alpha-1536)}{256(\alpha+1)}f_1^4. \qquad (5.56)
\end{aligned}
$$

Seeking a progressive wave solution for the equation (5.55) of the form $f_3 = f_3(\zeta_+)$, the following equation is obtained

$$
\frac{(\alpha+1)}{24}Af_3''' + \frac{(\alpha+1)}{2}(f_1f_3)' - \frac{(\alpha+1)}{6}Af_3' = R_4'(f_1). \qquad (5.57)
$$

We integrate (5.57) with respect to ζ_+ and use the localization condition to obtain

$$
\frac{(\alpha+1)}{24}Af_3'' + \frac{(\alpha+1)}{6}(3f_1 - A)f_3 = R_4(f_1). \qquad (5.58)
$$

Since the first term in the right-hand side of (5.56) cause to secularity in the solution of f_3, the coefficient of f_1 must vanish, which yields

$$
p_1(\tau) = \frac{5(\alpha+1)^3}{432}A^3\tau. \qquad (5.59)
$$

We propose a solution for f_3 of the following form

$$
f_3 = c_1 \operatorname{sech}^6 \zeta_+ + c_2 \operatorname{sech}^4 \zeta_+ + c_3 \operatorname{sech}^2 \zeta_+, \qquad (5.60)
$$

where c_i are constants to be determined from the solution of (5.58). Carrying out the derivative of f_3 we obtain

$$
\begin{aligned}
f_3'' = &- 42c_1 \operatorname{sech}^8 \zeta_+ + (36c_1 - 20c_2)\operatorname{sech}^6 \zeta_+\\
&+ (16c_2 - 6c_3)\operatorname{sech}^4 \zeta_+ + 4c_3 \operatorname{sech}^2 \zeta_+. \qquad (5.61)
\end{aligned}
$$

Introducing (5.60) and (5.61) into (5.58) and setting the coefficients of $\operatorname{sech}^8 \zeta_+$,

$\operatorname{sech}^6 \zeta_+$ and $\operatorname{sech}^4 \zeta_+$ equal to zero, we have

$$
\begin{aligned}
c_1 =& \frac{341\alpha^2 + 1472\alpha + 1536}{320(\alpha+1)^2} A^3, \\
c_2 =& -\frac{(670\alpha^3 + 3263\alpha^2 + 6646\alpha + 10128)}{960(\alpha+1)^2} A^3 + \frac{3(\alpha-3)^2}{16} A^2 B, \\
c_3 =& \frac{(230\alpha^4 + 1010\alpha^3 + 4999\alpha^2 + 2938\alpha + 9654)}{1440(\alpha+1)^2} A^3 \\
& - \frac{(\alpha-3)^2}{8} A^2 B.
\end{aligned} \tag{5.62}
$$

Then we can give the particular solution of the differential equation (5.58) by

$$
\begin{aligned}
f_3 =& \frac{(341\alpha^2 + 1472\alpha + 1536)}{320(\alpha+1)^2} f_1^3 \\
& - \left[\frac{(670\alpha^3 + 3263\alpha^2 + 6646\alpha + 10128)}{960(\alpha+1)^2} + \frac{3(\alpha-3)^2}{16} \frac{B}{A} \right] A f_1^2 \\
& + \left[\frac{(230\alpha^4 + 1010\alpha^3 + 4999\alpha^2 + 2938\alpha + 9654)}{1440(\alpha+1)^2} \right. \\
& \left. - \frac{(\alpha-3)^2}{8} \frac{B}{A} \right] A^2 f_1.
\end{aligned} \tag{5.63}
$$

Similar solution for the function $g_3(\zeta_-)$ can be given as

$$
\begin{aligned}
g_3 =& \frac{(341\alpha^2 + 1472\alpha + 1536)}{320(\alpha+1)^2} g_1^3 \\
& + \left[\frac{(670\alpha^3 + 3263\alpha^2 + 6646\alpha + 10128)}{960(\alpha+1)^2} + \frac{3(\alpha-3)^2}{16} \frac{A}{B} \right] B g_1^2 \\
& + \left[\frac{(230\alpha^4 + 1010\alpha^3 + 4999\alpha^2 + 2938\alpha + 9654)}{1440(\alpha+1)^2} \right. \\
& \left. + \frac{(\alpha-3)^2}{8} \frac{A}{B} \right] B^2 g_1, \qquad q_1(\tau) = -\frac{5(\alpha+1)^3}{432} B^3 \tau.
\end{aligned} \tag{5.64}
$$

Then the final solution of u_3 and S_3 read as

$$
\begin{aligned}
u_3 =& \frac{(621\alpha^2 + 2392\alpha + 2176)}{320(\alpha+1)^2} (f_1^3 + g_1^3) + \frac{3(\alpha-3)^2}{16} (A g_1^2 - B f_1^2) \\
& + \frac{(\alpha-3)^2}{8} AB(g_1 - f_1) + P_0 \frac{\partial f_1}{\partial \xi} + Q_0 \frac{\partial g_1}{\partial \eta} \\
& - \frac{(1100\alpha^3 + 4723\alpha^2 + 10436\alpha + 12888)}{960(\alpha+1)^2} (A f_1^2 - B g_1^2)
\end{aligned}
$$

$$+\frac{(300\alpha^4+1300\alpha^3+5719\alpha^2+3788\alpha+10004)}{1440(\alpha+1)^2}(A^2f_1+B^2g_1)$$
$$+\frac{(10\alpha^3-30\alpha^2+6\alpha+46)}{16(\alpha+1)}\left[\frac{(A-B)}{3}f_1g_1-\frac{(f_1g_1^2+f_1^2g_1)}{2}\right]$$
$$+\left[\frac{(5\alpha^3-6\alpha^2-4\alpha-101)}{48(\alpha+1)}A\frac{\partial f_1}{\partial\xi}-\frac{(\alpha^3+\alpha^2-9\alpha+63)}{48(\alpha+1)}B\frac{\partial f_1}{\partial\xi}\right.$$
$$-\frac{(4\alpha^3+5\alpha^2-19\alpha-56)}{16(\alpha+1)}f_1\frac{\partial f_1}{\partial\xi}+\frac{(2\alpha^3-\alpha^2-3\alpha-12)}{32(\alpha+1)}g_1\frac{\partial f_1}{\partial\xi}$$
$$\left.-\frac{(\alpha-3)^2}{16}f_1\frac{\partial g_1}{\partial\eta}\right]M+\left[\frac{(\alpha^3+\alpha^2-9\alpha+63)}{48(\alpha+1)}A\frac{\partial g_1}{\partial\eta}\right.$$
$$-\frac{(5\alpha^3-6\alpha^2-4\alpha-101)}{48(\alpha+1)}B\frac{\partial g_1}{\partial\eta}-\frac{(4\alpha^3+5\alpha^2-19\alpha-56)}{16(\alpha+1)}$$
$$\left.\times g_1\frac{\partial g_1}{\partial\eta}+\frac{(2\alpha^3-\alpha^2-3\alpha-12)}{32(\alpha+1)}f_1\frac{\partial g_1}{\partial\eta}-\frac{(\alpha-3)^2}{16}g_1\frac{\partial f_1}{\partial\xi}\right]N, \qquad (5.65)$$

$$S_3=\frac{(61\alpha^2+552\alpha+896)}{320(\alpha+1)^2}(f_1^3-g_1^3)-\frac{3(\alpha-3)^2}{16}(Ag_1^2+Bf_1^2)$$
$$-\frac{(\alpha-3)^2}{8}AB(f_1+g_1)+P_0\frac{\partial f_1}{\partial\xi}-Q_0\frac{\partial g_1}{\partial\eta}$$
$$-\frac{(80\alpha^3+601\alpha^2+952\alpha+2456)}{320(\alpha+1)^2}(Af_1^2+Bg_1^2)$$
$$+\frac{(160\alpha^4+720\alpha^3+4279\alpha^2+2088\alpha+9304)}{1440(\alpha+1)^2}(A^2f_1-B^2g_1)$$
$$+\frac{(9\alpha^3-26\alpha^2-1)}{16(\alpha+1)}(f_1g_1^2-f_1^2g_1)$$
$$+\frac{(7\alpha^3-22\alpha^2+6\alpha-73)}{24(\alpha+1)}(A+B)f_1g_1$$
$$+\left[\frac{(3\alpha^3-4\alpha^2+6\alpha-95)}{48(\alpha+1)}A\frac{\partial f_1}{\partial\xi}-\frac{(\alpha^3+\alpha^2-9\alpha+63)}{48(\alpha+1)}B\frac{\partial f_1}{\partial\xi}\right.$$
$$-\frac{(4\alpha^3-3\alpha^2-3\alpha-32)}{16(\alpha+1)}f_1\frac{\partial f_1}{\partial\xi}+\frac{(6\alpha^3-21\alpha^2+9\alpha+24)}{32(\alpha+1)}$$
$$\left.\times g_1\frac{\partial f_1}{\partial\xi}+\frac{(\alpha-3)^2}{16}f_1\frac{\partial g_1}{\partial\eta}\right]M+\left[-\frac{(\alpha^3+\alpha^2-9\alpha+63)}{48(\alpha+1)}A\frac{\partial g_1}{\partial\eta}\right.$$
$$+\frac{(3\alpha^3-4\alpha^2+6\alpha-95)}{48(\alpha+1)}B\frac{\partial g_1}{\partial\eta}+\frac{(4\alpha^3-3\alpha^2-3\alpha-32)}{16(\alpha+1)}$$
$$\left.\times g_1\frac{\partial g_1}{\partial\eta}-\frac{(6\alpha^3-21\alpha^2+9\alpha+24)}{32(\alpha+1)}f_1\frac{\partial g_1}{\partial\eta}-\frac{(\alpha-3)^2}{16}g_1\frac{\partial f_1}{\partial\xi}\right]N. \qquad (5.66)$$

Thus, for this order, the trajectories of the solitary waves become

$$\epsilon^{\frac{1}{2}}(x-t) = \xi + \epsilon p_0(\tau) + \epsilon^2 p_1(\tau) + \epsilon^2 P_0 + \mathcal{O}(\epsilon^3),$$
$$\epsilon^{\frac{1}{2}}(x+t) = \eta + \epsilon q_0(\tau) + \epsilon^2 q_1(\tau) + \epsilon^2 Q_0 + \mathcal{O}(\epsilon^3). \quad (5.67)$$

5.4.1 Phase Shifts

To obtain the phase shifts after a head-on collision of solitary waves characterized by A and B are asymptotically far from each other at the initial time ($t = -\infty$), the solitary wave A is at $\xi = 0$, $\eta = -\infty$, and the solitary wave B is at $\eta = 0$, $\xi = +\infty$, respectively. After the collision ($t = +\infty$), the solitary wave B is far to the right of solitary wave A, i.e., the solitary wave A is at $\xi = 0$, $\eta = +\infty$, and the solitary wave B is at $\eta = 0$, $\xi = -\infty$. Using the equations (5.20) and (5.52) one can obtain the corresponding phase shifts Δ_A and Δ_B as follows:

$$\begin{aligned}
\Delta_A &= \epsilon^{1/2}(x-t)\mid_{\xi=0,\eta=\infty} - \epsilon^{1/2}(x-t)\mid_{\xi=0,\eta=-\infty} \\
&= \epsilon^2\left(\frac{2\alpha^2-5\alpha+2}{8}\right) f_1(0)\int_{-\infty}^{+\infty} g_1(\eta')d\eta' \\
&= \epsilon^2\left(\frac{2\alpha^2-5\alpha+2}{8}\right) A\int_{-\infty}^{+\infty} g_1(\eta')d\eta' \\
&= -\epsilon^2\left(\frac{2\alpha^2-5\alpha+2}{4}\right)\left(\frac{12}{\alpha+1}\right)^{1/2} AB^{1/2},
\end{aligned} \quad (5.68)$$

$$\begin{aligned}
\Delta_B &= \epsilon^{1/2}(x+t)\mid_{\eta=0,\xi=-\infty} - \epsilon^{1/2}(x+t)\mid_{\eta=0,\xi=\infty} \\
&= -\epsilon^2\left(\frac{2\alpha^2-5\alpha+2}{8}\right) g_1(0)\int_{-\infty}^{+\infty} f_1(\xi')d\xi' \\
&= \epsilon^2\left(\frac{2\alpha^2-5\alpha+2}{8}\right) B\int_{-\infty}^{+\infty} f_1(\xi')d\xi' \\
&= \epsilon^2\left(\frac{2\alpha^2-5\alpha+2}{4}\right)\left(\frac{12}{\alpha+1}\right)^{1/2} A^{1/2}B.
\end{aligned} \quad (5.69)$$

Here, as opposed to the results of previous works on the same subject the phase shifts depend on the amplitudes of both waves.

5.5 Summary of the results

In the previous section, we have obtained the following results

$$
\begin{aligned}
u(f_1,g_1) =& \epsilon\Bigg\{(f_1+g_1)+\epsilon\left[-\frac{(11\alpha+20)}{8(\alpha+1)}(f_1^2-g_1^2)\right.\\
&+\frac{(6\alpha^2+13\alpha+34)}{12(\alpha+1)}(Af_1+Bg_1)+\frac{(\alpha-3)}{4}\Bigg(M(\eta,\tau)\frac{\partial f}{\partial\xi}\\
&\left.-N(\xi,\tau)\frac{\partial g}{\partial\eta}\Bigg)\right]+\epsilon^2\left[\frac{(621\alpha^2+2392\alpha+2176)}{320(\alpha+1)^2}(f_1^3+g_1^3)\right.\\
&+\frac{3(\alpha-3)^2}{16}(Ag_1^2-Bf_1^2)+\frac{(\alpha-3)^2}{8}AB(g_1-f_1)+P_0\\
&\times\frac{\partial f_1}{\partial\xi}+Q_0\frac{\partial g_1}{\partial\eta}-\frac{(1100\alpha^3+4723\alpha^2+10436\alpha+12888)}{960(\alpha+1)^2}\\
&\times(Af_1^2-Bg_1^2)\\
&+\frac{(300\alpha^4+1300\alpha^3+5719\alpha^2+3788\alpha+10004)}{1440(\alpha+1)^2}(A^2f_1\\
&+B^2g_1)+\frac{(10\alpha^3-30\alpha^2+6\alpha+46)}{16(\alpha+1)}\Bigg(\frac{(A-B)}{3}f_1g_1\\
&-\frac{(f_1g_1^2+f_1^2g_1)}{2}\Bigg)+\Bigg(\frac{(5\alpha^3-6\alpha^2-4\alpha-101)}{48(\alpha+1)}A\frac{\partial f_1}{\partial\xi}\\
&-\frac{(\alpha^3+\alpha^2-9\alpha+63)}{48(\alpha+1)}B\frac{\partial f_1}{\partial\xi}-\frac{(4\alpha^3+5\alpha^2-19\alpha-56)}{16(\alpha+1)}\\
&\times f_1\frac{\partial f_1}{\partial\xi}+\frac{(2\alpha^3-\alpha^2-3\alpha-12)}{32(\alpha+1)}g_1\frac{\partial f_1}{\partial\xi}\\
&-\frac{(\alpha-3)^2}{16}f_1\frac{\partial g_1}{\partial\eta}\Bigg)M+\Bigg(\frac{(\alpha^3+\alpha^2-9\alpha+63)}{48(\alpha+1)}A\frac{\partial g_1}{\partial\eta}\\
&-\frac{(5\alpha^3-6\alpha^2-4\alpha-101)}{48(\alpha+1)}B\frac{\partial g_1}{\partial\eta}\\
&-\frac{(4\alpha^3+5\alpha^2-19\alpha-56)}{16(\alpha+1)}g_1\frac{\partial g_1}{\partial\eta}
\end{aligned}
$$

$$+\frac{(2\alpha^3-\alpha^2-3\alpha-12)}{32(\alpha+1)}f_1\frac{\partial g_1}{\partial\eta}-\frac{(\alpha-3)^2}{16}g_1\frac{\partial f_1}{\partial\xi}\Big)N\Big]$$

$$+\ldots\Big\} \tag{5.70}$$

Similar expression may be given for $S(f_1, g_1)$.

$$p(\tau)=\epsilon\left(\frac{(\alpha+1)^2}{24}A^2+\epsilon\frac{5(\alpha+1)^3}{432}A^3\right)\tau, \tag{5.71}$$

$$q(\tau)=\epsilon\left(-\frac{(\alpha+1)^2}{24}B^2-\epsilon\frac{5(\alpha+1)^3}{432}B^3\right)\tau, \tag{5.72}$$

$$P=\frac{(2\alpha^2-5\alpha+2)}{8}f_1(\xi,\tau)\int_{-\infty}^{\eta}g_1(\eta',\tau)d\eta', \tag{5.73}$$

$$Q=\frac{(2\alpha^2-5\alpha+2)}{8}g_1(\eta,\tau)\int_{\infty}^{\xi}f_1(\xi',\tau)d\xi', \tag{5.74}$$

and

$$\zeta_+=\left[\frac{(\alpha+1)A\epsilon}{12}\right]^{\frac{1}{2}}\Bigg(x-c_Rt-\epsilon^{\frac{3}{2}}\frac{(2\alpha^2-5\alpha+2)}{8}f_1(\xi,\tau)$$

$$\times\int_{-\infty}^{\eta}g_1(\eta',\tau)d\eta'\Bigg), \tag{5.75}$$

$$\zeta_-=\left[\frac{(\alpha+1)B\epsilon}{12}\right]^{\frac{1}{2}}\Bigg(x+c_Lt-\epsilon^{\frac{3}{2}}\frac{(2\alpha^2-5\alpha+2)}{8}g_1(\eta,\tau)$$

$$\times\int_{\infty}^{\xi}f_1(\xi',\tau)d\xi'\Bigg), \tag{5.76}$$

where c_R and c_L are defined by

$$c_R=1+\left(\epsilon\frac{(\alpha+1)}{6}A+\epsilon^2\frac{(\alpha+1)^2}{24}A^2+\epsilon^3\frac{5(\alpha+1)^3}{432}A^3\right), \tag{5.77}$$

$$c_L=1+\left(\epsilon\frac{(\alpha+1)}{6}B+\epsilon^2\frac{(\alpha+1)^2}{24}B^2+\epsilon^3\frac{5(\alpha+1)^3}{432}B^3\right). \tag{5.78}$$

The equations (5.73) and (5.74) serve to define the phase changes. Before the collision

$$\eta \to -\infty, \quad P \to 0, \qquad \xi \to \infty, \quad Q \to 0 \tag{5.79}$$

and after the collision

$$\eta \to \infty, \qquad P = -\frac{(2\alpha^2 - 5\alpha + 2)}{2(\alpha+1)^{\frac{1}{2}}} A\,(3B)^{\frac{1}{2}} \operatorname{sech}^2 \zeta_+, \tag{5.80}$$

$$\xi \to -\infty, \qquad Q = \frac{(2\alpha^2 - 5\alpha + 2)}{2(\alpha+1)^{\frac{1}{2}}} B\,(3A)^{\frac{1}{2}} \operatorname{sech}^2 \zeta_-. \tag{5.81}$$

In this section we shall illustrate the profiles of right-going waves before and after the collision. For that purpose we set $g_1(\eta, \tau) = 0$ in the expression u and obtain

$$u = \epsilon \left\{ f_1 + \epsilon \left[-\frac{(11\alpha + 20)}{8(\alpha+1)} f_1^2 + \frac{(6\alpha^2 + 13\alpha + 34)}{12(\alpha+1)} A f_1 \right] \right. \tag{5.82}$$

$$+\epsilon^2 \left[\frac{(621\alpha^2 + 2392\alpha + 2176)}{320(\alpha+1)^2} f_1^3 \right.$$
$$- \left(\frac{(1100\alpha^3 + 4723\alpha^2 + 10436\alpha + 12888)}{960(\alpha+1)^2} + \frac{3(\alpha-3)^2 B}{16A} \right) A f_1^2$$
$$+ \left(\frac{(300\alpha^4 + 1300\alpha^3 + 5719\alpha^2 + 3788\alpha + 10004)}{1440(\alpha+1)^2} \right.$$
$$\left. \left. - \frac{(\alpha-3)^2 B}{8A} \right) A^2 f_1 \right] + \ldots \right\} \tag{5.83}$$

with

$$f_1 = A \operatorname{sech}^2 \left[\left(\frac{(\alpha+1)A\epsilon}{12} \right)^{\frac{1}{2}} (x - c_R t + \Theta) \right] \tag{5.84}$$

where

$$\Theta = \epsilon^{\frac{3}{2}} \frac{(2\alpha^2 - 5\alpha + 2)}{2(\alpha+1)^{\frac{1}{2}}} A\,(3B)^{\frac{1}{2}} \operatorname{sech}^2 \zeta_+. \tag{5.85}$$

The variations of the wave profiles for velocity parameter u before the collision ($\Theta = 0$) and after the collision (Θ is given as in (5.85)) are depicted in Figure 5.1, for various values of parameters ϵ, α, A and B. As is seen from the figure the wave profile before the collision is symmetric, whereas after the collision it is unsymmetrical and tilts backward with respect to the direction of its propagation.

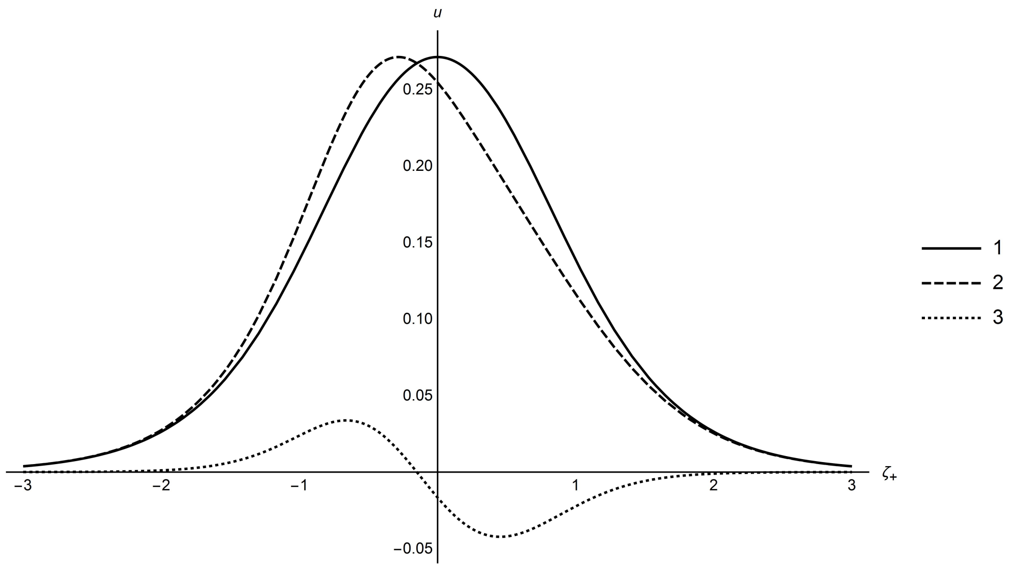

Figure 5.1: Right-going wave profile u for $\epsilon = 0.4$, $\alpha = 4$, $A = B = 0.5$. 1:before collision; 2:after collision; 3:difference between the wave profiles before and after the collision.

5.6 Result and Conclusion

Employing the non-dimensional field equations (5.1)-(5.3) and the extended PLK method, we have studied the head-on collision of solitary waves in arteries. Introducing a set of stretched coordinates that include some unknown functions characterizing the higher order dispersive effects and the trajectory functions, which are to be determined from the removal of possible secularities that might occur in the solution, expanding these unknown functions and the field variables into power series of the smallness parameter ϵ and introducing the resulting expansions into the field equations we obtained the sets of partial differential equations. By solving these differential equations and imposing the requirements for the removal of possible secularities we obtained the speed correction terms and the trajectory functions. The results of our calculation show that both the evolution equations and the phase shifts resulting from the head-on collision of solitary waves are quite different from those of Xue [7], who employed the incorrect formulation of Su and Mirie [6]. As opposed to the result of previous works on the same subject, in the present work the phase shifts depend on the amplitudes of both colliding waves.

Chapter 6

Conclusion

In this study, head-on collision problem between two solitary waves was examined in two different media which are: the shallow water and the fluid-filled elastic tube. First, the historical background of solitary waves was presented and the previous experimental, numerical and theoretical studies about solitary wave interactions were reviewed. The necessity of employing some kind of numerical or asymptotic method to study the head-on collision problem between two solitary waves was indicated. The derivation of the field equations for shallow water waves and fluid-filled elastic tubes was summarized and then the reductive perturbation method and Poincaré-Lighthill-Kuo (PLK) method were briefly reviewed. Su and Mirie's [6] perturbation approach to the head-on collision problem was mentioned and the observation of the incorrectness of their statement about the secular terms was declared (The derivation of their study was also presented in Appendix A).

Based on this observation, the head-on collision problem between two solitary waves in shallow water was re-examined by introducing a set of stretched coordinates in which the trajectory functions were of order ϵ^2. The evolution equations governing the colliding waves were obtained under the non-secularity conditions and the progressive wave solutions to these equations were provided. Also, the trajectory functions were determined by using the restrictions that had been resulted from the elimination of the secular terms. Explicit expressions of the phase shifts of right and left going waves were obtained and the following results were concluded:

(i) Phase shifts are found to be depend on amplitudes of both colliding waves in contrast to the result of previous studies.

(ii) Order of phase shifts is ϵ^2 rather than ϵ.

In the next chapter, the head-on collision between two solitary waves was examined by using the extended PLK method. Since a different set of stretched coordinates was introduced, the evolution equations for various order of ϵ were obtained as KdV equation and linearized KdV equation with nonhomogeneous term rather than ordinary differential equations. By providing progressive wave solutions to the evolution equations and imposing the requirements for the removal of secularities, the speed correction terms and the trajectory functions were obtained. It was concluded that the results presented here were same with the results found in the previous chapter whereas they were totally different from the results of Su and Mirie [6]. Also, the variations of the wave profiles for right going wave before and after collision were illustrated. It was observed that the wave profile was symmetric before the collision whereas it was unsymmetrical and tilted backwards with respect to the direction of its propagation after the collision.

In the final part of the study, head-on collision of the solitary waves in fluid-filled elastic tubes was examined by employing the extended PLK method. Evolution equations were obtained as KdV equation and linearized KdV equation with nonhomogeneous term. By seeking a progressive wave solution to evolution equations, the speed correction terms and the trajectory functions were obtained. As a consequence of our calculations, the following results were concluded:

(i) Both the evolution equations and the phase shifts are quite different from those of Xue [7], who employed the incorrect formulation of Su and Mirie [6].

(ii) As opposed to the results of previous studies on the same subject, the order of the trajectory functions is ϵ^2 rather than ϵ and the phase shifts depend on the amplitude of both colliding waves.

The variations of the wave profiles before and after the collision were depicted on some figures and it was seen that the wave profile was symmetric before the collision whereas it was unsymmetrical and tilted backwards with respect to the direction of its propagation after the collision.

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Appendix A

Derivation of Su and Mirie (1980) Article

In their pioneering work Su and Mirie [6] studied the head-on collision of two solitary waves in shallow water by introducing a set of stretched coordinates

$$
\begin{aligned}
\epsilon^{\frac{1}{2}} k(x - C_R t) &= \xi - \epsilon k \theta(\xi, \eta), \\
\epsilon^{\frac{1}{2}} l(x + C_L t) &= \eta - \epsilon l \phi(\xi, \eta),
\end{aligned} \tag{A.1}
$$

where ϵ is the smallness parameter measuring the order of nonlinearity, C_R and C_L are the speeds of right and left going waves, k and l are the wave numbers of order unity of the corresponding waves, $\theta(\xi, \eta)$ and $\phi(\xi, \eta)$ are some unknown functions describing the trajectories of the right and left going waves and they will be used in obtaining phase shifts.

Employing the transformation (A.1) in a shallow water theory with constant depth, Su and Mirie [6] studied the head-on collision of two solitary waves by reducing the fluid equations and the boundary equations to the following coupled differential equations,

$$
\begin{aligned}
&2\epsilon(C_R + C_L)\left[l\frac{\partial \alpha}{\partial \eta} + \epsilon k l\left(\frac{\partial \theta}{\partial \eta}\frac{\partial \alpha}{\partial \xi} - \frac{\partial \theta}{\partial \xi}\frac{\partial \alpha}{\partial \eta}\right)\right] + \left\{k\frac{\partial}{\partial \xi} + l\frac{\partial}{\partial \eta} + \right. \\
&\left.\epsilon k l\left[\frac{\partial}{\partial \eta}(\theta - \phi)\frac{\partial}{\partial \xi} - \frac{\partial}{\partial \xi}(\theta - \phi)\frac{\partial}{\partial \eta}\right]\right\} F_+ = 0,
\end{aligned} \tag{A.2}
$$

$$
\begin{aligned}
&2\epsilon(C_L + C_R)\left[k\frac{\partial \beta}{\partial \xi} + \epsilon l k\left(\frac{\partial \phi}{\partial \xi}\frac{\partial \beta}{\partial \eta} - \frac{\partial \phi}{\partial \eta}\frac{\partial \beta}{\partial \xi}\right)\right] + \left\{l\frac{\partial}{\partial \eta} + k\frac{\partial}{\partial \xi} + \right. \\
&\left.\epsilon l k\left[\frac{\partial}{\partial \xi}(\phi - \eta)\frac{\partial}{\partial \eta} - \frac{\partial}{\partial \eta}(\phi - \theta)\frac{\partial}{\partial \xi}\right]\right\} F_- = 0,
\end{aligned} \tag{A.3}
$$

where the expressions of F_+, F_-, α and β were defined as follows:

$$F_{\pm} = \pm\,(1 - C_{R,L})(w \pm \zeta) + \frac{w^2}{2} \pm \zeta w + \sum_{n=1}^{\infty}(-1)^n \frac{(1+\zeta)^{2n}}{(2n)!}$$
$$\times \left[\frac{\partial^{2n} w}{\partial t \partial x^{2n-1}} \pm \frac{(1+\zeta)}{2n+1}\frac{\partial^{2n} w}{\partial x^{2n}} + \frac{1}{2}\sum_{m=0}^{2n}(-1)^m \binom{2n}{m}\frac{\partial^m w}{\partial x^m}\right.$$
$$\left.\times \frac{\partial^{2n-m} w}{\partial x^{2n-m}}\right], \tag{A.4}$$

$$w + \zeta = 2\epsilon\alpha, \qquad w - \zeta = -2\epsilon\beta. \tag{A.5}$$

They expanded the field variables into the following power series of ϵ,

$$\begin{aligned}
\alpha(\xi,\eta) &= \alpha_0 + \epsilon\alpha_1 + \epsilon^2\alpha_2 + \ldots\ ,\\
\beta(\xi,\eta) &= \beta_0 + \epsilon\beta_1 + \epsilon^2\beta_2 + \ldots\ ,\\
\theta(\xi,\eta) &= \theta_0(\eta) + \epsilon\theta_1(\xi,\eta) + \ldots\ ,\\
\phi(\xi,\eta) &= \phi_0(\xi) + \epsilon\phi_1(\xi,\eta) + \ldots\ ,\\
C_R &= 1 + \epsilon a R_1 + \epsilon^2 a^2 R_2 + \epsilon^3 a^3 R_3 + \ldots\ ,\\
C_L &= 1 + \epsilon b L_1 + \epsilon^2 b^2 L_2 + \epsilon^3 b^3 L_3 + \ldots\ .
\end{aligned} \tag{A.6}$$

By introducing the expansion (A.6) into the field equations (A.2) and (A.3) and setting the coefficients of like powers of ϵ equal to zero, they obtained a set of differential equations governing α_0, α_1, α_2, ..., θ_0, θ_1, θ_2, ..., (β_0, β_1, β_2, ..., ϕ_0, ϕ_1, ϕ_2, ...), the first two of them are given as follows:

$O(\epsilon)$ *equations:*

$$\frac{\partial \alpha_0}{\partial \eta} = 0, \quad \frac{\partial \beta_0}{\partial \xi} = 0.$$

The solution of these equations are

$$\alpha_0 = a f(\xi), \quad \beta_0 = b g(\eta) \tag{A.7}$$

where a and b are the amplitudes of the corresponding waves.

$O\left(\epsilon^2\right)$ *equations:*

$$4l\left(\frac{\partial \alpha_1}{\partial \eta} + ak\frac{\partial \theta_0}{\partial \eta}f^{'}\right) - 2R_1a^2kf^{'} + 3a^2kff^{'} - b^2lgg^{'} - ablfg^{'}$$
$$- abkf^{'}g + \frac{1}{3}\left(ak^3f^{'''} + 2bl^3g^{'''}\right) = 0 \quad (A.8)$$

where a prime denotes the differentiation of the corresponding quantities with respect to their arguments. A similar expression may be given for β_1 provided that proper substitutions are made in the variables.

Integrating (A.8) with respect to η we obtain

$$4l\alpha_1 + \eta\left(\frac{ak^3}{3}f^{'''} + 3a^2kff^{'} - 2R_1a^2kf^{'}\right) + ak\left[4l\theta_0 - bM(\eta)\right]f^{'}$$
$$- b^2l\frac{g^2}{2} - ablfg + \frac{2}{3}bl^3g^{''} = 4lF_1(\xi) \quad (A.9)$$

where $M(\eta)$ is defined by

$$M(\eta) = \int^{\eta} g(\eta^{'})d\eta^{'}. \quad (A.10)$$

In equation (A.9), as $\eta \to \pm\infty$ the second term causes the secularity and, thus, the coefficient of η must vanish

$$\frac{ak^3}{3}f^{'''} + 3a^2kff^{'} - 2R_1a^2kf^{'} = 0. \quad (A.11)$$

This is the evolution equation for the right going wave and a similar expression may be given for left going wave. By letting $k^2 = 3a$ and $R_1 = 1/2$, the solution of the equation (A.11) may be given by

$$f = \text{sech}^2\left(\frac{\xi}{2}\right). \quad (A.12)$$

Similarly, for right going wave we have

$$L_1 = \frac{1}{2}, \quad l^2 = 3b, \quad g = \text{sech}^2\left(\frac{\eta}{2}\right). \quad (A.13)$$

For the solutions of the type (A.12) and (A.13), the functions $M(\eta)$ and $N(\xi)$ will be of the form $\tanh\left(\frac{\eta}{2}\right)$ and $\tanh\left(\frac{\xi}{2}\right)$, respectively. The integral of them, that is, $\int^{\eta} M(\eta^{'})d\eta^{'}$ and $\int^{\xi} N(\xi^{'})d\xi^{'}$ approaches to infinity as $\xi(\eta) \to \pm\infty$. Su and Mirie [6] made the statement that "although the third term in equation (A.9) does not lead to secularity for this order it may cause the secularity in the higher order terms, therefore the coefficient of $f^{'}$ in the third term must vanish"

$$4l\theta_0 - b\int_{-\infty}^{\eta} g(\eta^{'})d\eta^{'} = 0. \tag{A.14}$$

This equation makes it possible to determine the unknown trajectory function θ_0. Unfortunately, our calculations for the higher order terms show that the term $M(\eta)f^{'}$ does not cause any secularity in the next order. Our result is also justified by one of the authors of the paper (Dr. Su). To see this is the case, we keep the terms $M(\eta) = \int^{\eta} g(\eta^{'})d\eta^{'}$ and $N(\xi) = \int^{\xi} f(\xi^{'})d\xi^{'}$ in the expression of α_1 and β_1, respectively. Then we have

$$\begin{aligned}\alpha_1 &= \frac{7}{8}b^2g^2 + \frac{ab}{4}fg - \frac{b^2}{2}g + \frac{abk}{4l}f^{'}M(\eta) + a^2F_1(\xi),\\ \beta_1 &= \frac{7}{8}a^2f^2 + \frac{ab}{4}fg - \frac{a^2}{2}f + \frac{abl}{4k}g^{'}N(\xi) + b^2G_1(\eta).\end{aligned} \tag{A.15}$$

Now if we substitute (A.7) and (A.15) into the next order equation, then we obtain the coefficient of the term $\int^{\eta} M(\eta^{'})d\eta^{'}$ in the expression of α_2 as follows

$$\alpha_2 = -\frac{a^2bk^2}{16l^2}\left(f^{(iv)} + 3ff^{''} + 3(f^{'})^2 - f^{''}\right)\int^{\eta} M(\eta^{'})d\eta^{'} + \dots. \tag{A.16}$$

Keeping in mind that $k^2 = 3a$ and $R_1 = 1/2$, the coefficient of the secular term in (A.16) is nothing but the derivative of the evolution equation (A.11) with respect to its argument; thus, the coefficient of the secular term vanishes. Therefore, the statement made by Su and Mirie is not correct and θ_0 remains undetermined and it should be set equal to zero. This shows that in the transformation (A.1) the order of $\theta(\xi,\eta)$ and $\phi(\xi,\eta)$ must be of order ϵ^2 rather than ϵ.

Appendix B

$\mathcal{O}(\epsilon^4)$ Equations for Chapter 3

$$
\begin{aligned}
&4l\frac{\partial \alpha_3}{\partial \eta} + ak\frac{\partial^3}{\partial \xi^3}(\alpha_2 - \beta_2) - 3bk\frac{\partial^3}{\partial \xi \partial \eta^2}(\alpha_2 - \beta_2) - 2bl\frac{\partial^3}{\partial \eta^3}(\alpha_2 - \beta_2) \\
&- ak\frac{\partial \alpha_2}{\partial \xi} + bl\frac{\partial \alpha_2}{\partial \eta} + 3k\frac{\partial}{\partial \xi}(\alpha_0\alpha_2) - k\beta_0\frac{\partial \alpha_2}{\partial \xi} - l\frac{\partial}{\partial \eta}(\beta_0\alpha_2) \\
&- k\frac{\partial}{\partial \xi}(\alpha_0\beta_2) - k\beta_0\frac{\partial \beta_2}{\partial \xi} - l(\alpha_0 + \beta_0)\frac{\partial \beta_2}{\partial \eta} - l\frac{\partial \beta_0}{\partial \eta}\beta_2 + 3l\alpha_0\frac{\partial \alpha_2}{\partial \eta} \\
&- \frac{3}{10}a^2k\frac{\partial^5}{\partial \xi^5}(\alpha_1 - \beta_1) - \frac{3}{4}a^2l\frac{\partial^5}{\partial \xi^4 \partial \eta}(\alpha_1 - \beta_1) + \frac{3}{2}abl\frac{\partial^5}{\partial \xi^2 \partial \eta^3}(\alpha_1 \\
&- \beta_1) + \frac{3}{2}b^2k\frac{\partial^5}{\partial \xi \partial \eta^4}(\alpha_1 - \beta_1) + \frac{9}{20}b^2l\frac{\partial^5}{\partial \eta^5}(\alpha_1 - \beta_1) + \left(\frac{3}{4}a^2k\right. \\
&\left.+3ak\beta_0\right)\frac{\partial^3}{\partial \xi^3}(\alpha_1 - \beta_1) + \left(\frac{3}{2}a^2l - \frac{3}{4}abl - 6al\alpha_0 + 3al\beta_0\right) \\
&\times \frac{\partial^3}{\partial \xi^2 \partial \eta}(\alpha_1 - \beta_1) + \left(\frac{3}{4}abk - \frac{3}{2}b^2k - 12bk\alpha_0 - 3bk\beta_0\right) \\
&\times \frac{\partial^3}{\partial \xi \partial \eta^2}(\alpha_1 - \beta_1) - \left(\frac{3}{4}b^2l - 6bl\alpha_0 - 3bl\beta_0\right)\frac{\partial^3}{\partial \eta^3}(\alpha_1 - \beta_1) \\
&+ 3ak\alpha_0'\frac{\partial^2}{\partial \xi^2}(\alpha_1 - \beta_1) - 6bk\beta_0'\frac{\partial^2}{\partial \xi \partial \eta}(\alpha_1 - \beta_1) - \left(3bk\alpha_0' + 6bl\beta_0'\right) \\
&\times \frac{\partial^2}{\partial \eta^2}(\alpha_1 - \beta_1) + \left(3ak\alpha_0'' + 3bk\beta_0'' + 4kl\frac{\partial \theta_1}{\partial \eta} - \frac{19}{20}a^2k\right)\frac{\partial \alpha_1}{\partial \xi} \\
&+ \left(3al\alpha_0'' + 3bl\beta_0'' - 4kl\frac{\partial \theta_1}{\partial \xi} + \frac{19}{20}b^2l\right)\frac{\partial \alpha_1}{\partial \eta} + 6bk\beta_0''\frac{\partial \beta_1}{\partial \xi}
\end{aligned}
$$

$$
\begin{aligned}
&+ 6bl\beta_0^{''}\frac{\partial \beta_1}{\partial \eta} - k\frac{\partial}{\partial \xi}(\alpha_1\beta_1) - l\frac{\partial}{\partial \eta}(\alpha_1\beta_1) + \frac{1}{2}k\frac{\partial}{\partial \xi}(3\alpha_1^2 - \beta_1^2) \\
&+ \frac{1}{2}l\frac{\partial}{\partial \eta}(3\alpha_1^2 - \beta_1^2) + 3bl\beta_0^{'''}(2\alpha_1 + \beta_1) + 3ak\alpha_0^{'''}\beta_1 + \frac{9}{280}a^3k\alpha_0^{(vii)} \\
&+ \frac{3}{70}b^3l\beta_0^{(vii)} - \left(\frac{3}{16}a^3k + \frac{3}{4}a^2k\alpha_0 + \frac{3}{2}a^2k\beta_0\right)\alpha_0^{(v)} - \left(\frac{3}{16}b^3l\right. \\
&\left.+\frac{9}{4}b^2l\alpha_0 + \frac{3}{2}b^2l\beta_0\right)\beta_0^{(v)} - \frac{9}{4}a^2k\alpha_0^{'}\alpha_0^{(iv)} - \left(\frac{3}{4}b^2k\alpha_0^{'} + 3b^2l\beta_0^{'}\right)\beta_0^{(iv)} \\
&+ \left(\frac{57}{80}a^3k + \frac{3}{4}a^2k\alpha_0^{''} - \frac{3}{4}abk\beta_0^{''} - 3ak(\alpha_0^2 - \beta_0^2) + \frac{3}{2}a^2k(\alpha_0 + \beta_0)\right. \\
&\left.-3akl\frac{\partial \phi_1}{\partial \eta}\right)\alpha_0^{'''} + \left(\frac{57}{80}b^3l + \frac{3}{4}b^2l\beta_0^{''} - \frac{3}{4}abl\alpha_0^{''} + 6bl\alpha_0^2 + \frac{3}{2}b^2l\right. \\
&\left.\times(\alpha_0 + \beta_0) + 6bl\alpha_0\beta_0 - 3bkl\frac{\partial}{\partial \xi}(2\theta_1 - \phi_1)\right)\beta_0^{'''} + \left(\frac{3}{2}a^2k\alpha_0^{'}\right. \\
&\left.+\frac{3}{2}a^2l\beta_0^{'} + 6ak\alpha_0^{'}\beta_0 - 6al\alpha_0\beta_0^{'} - 3al^2\frac{\partial^2 \theta_1}{\partial \eta^2} - 3akl\frac{\partial^2 \phi_1}{\partial \xi \partial \eta}\right)\alpha_0^{''} \\
&+ \left(\frac{3}{2}b^2k\alpha_0^{'} + \frac{3}{2}b^2l\beta_0^{'} + 6bk\alpha_0\alpha_0^{'} + 12bl\alpha_0\beta_0^{'} + 6bl\beta_0\beta_0^{'} - 3al^2\frac{\partial^2 \theta_1}{\partial \xi^2}\right. \\
&\left.-3bkl\frac{\partial^2}{\partial \xi \partial \eta}(2\theta_1 - \phi_1)\right)\beta_0^{''} + 3ak(\alpha_0^{'})^3 + 3bl(\beta_0^{'})^3 - 3al\beta_0^{'}(\alpha_0^{'})^2 \\
&- 3bk\alpha_0^{'}(\beta_0^{'})^2 + \left(akl\frac{\partial^3}{\partial \xi^2 \partial \eta}(\theta_1 - \phi_1) - al^2\frac{\partial^3}{\partial \xi \partial \eta^2}(\theta_1 - \phi_1) - 2bkl\right. \\
&\left.\times\frac{\partial^3}{\partial \eta^3}(\theta_1 - \phi_1) + kl(3\alpha_0 - \beta_0)\frac{\partial}{\partial \eta}(\theta_1 - \phi_1) + bkl\frac{\partial \theta_1}{\partial \eta} + akl\frac{\partial \phi_1}{\partial \eta}\right)\alpha_0^{'} \\
&- \left(al^2\frac{\partial^3}{\partial \xi^2 \partial \eta}(\theta_1 - \phi_1) + 2bkl\frac{\partial^3}{\partial \xi \partial \eta^2}(\theta_1 - \phi_1) - kl(\alpha_0 + \beta_0)\right. \\
&\left.\times\frac{\partial}{\partial \xi}(\theta_1 - \phi_1)\right)\beta_0^{'} + 4kl\frac{\partial \theta_2}{\partial \eta}\alpha_0^{'} - 2R_3a^3k\alpha_0^{'} = 0 \qquad \text{(B.1)}
\end{aligned}
$$

A similar equation can be given for the variable β_3.

Appendix C

$\mathcal{O}(\epsilon^4)$ Equations for Chapter 4

$$
\begin{aligned}
&\frac{\partial \zeta_3}{\partial \eta} - \frac{\partial \zeta_3}{\partial \xi} + \frac{\partial w_3}{\partial \eta} + \frac{\partial w_3}{\partial \xi} + \frac{\partial \zeta_2}{\partial \tau} + \frac{\partial}{\partial \eta}(\zeta_2 w_0) + \frac{\partial}{\partial \xi}(\zeta_2 w_0) \\
&+ \frac{\partial}{\partial \eta}(\zeta_0 w_2) + \frac{\partial}{\partial \xi}(\zeta_0 w_2) + \frac{\partial}{\partial \eta}(\zeta_1 w_1) + \frac{\partial}{\partial \xi}(\zeta_1 w_1) \\
&- \frac{1}{6}\left(\frac{\partial^3 w_2}{\partial \xi^3} + 3\frac{\partial^3 w_2}{\partial \xi^2 \partial \eta} + 3\frac{\partial^3 w_2}{\partial \xi \partial \eta^2} + \frac{\partial^3 w_2}{\partial \eta^3}\right) - \frac{dq_0}{d\tau}\frac{\partial \zeta_1}{\partial \eta} - \frac{dp_0}{d\tau}\frac{\partial \zeta_1}{\partial \xi} \\
&+ \frac{1}{120}\left(\frac{\partial^5 w_1}{\partial \xi^5} + 5\frac{\partial^5 w_1}{\partial \xi^4 \partial \eta} + 10\frac{\partial^5 w_1}{\partial \xi^3 \partial \eta^2} + 10\frac{\partial^5 w_1}{\partial \xi^2 \partial \eta^3} + 5\frac{\partial^5 w_1}{\partial \xi \partial \eta^4}\right. \\
&\left.+\frac{\partial^5 w_1}{\partial \eta^5}\right) - \frac{\zeta_0}{2}\left(\frac{\partial^3 w_1}{\partial \xi^3} + 3\frac{\partial^3 w_1}{\partial \xi^2 \partial \eta} + 3\frac{\partial^3 w_1}{\partial \xi \partial \eta^2} + \frac{\partial^3 w_1}{\partial \eta^3}\right) - \frac{\zeta_1}{2}\left(\frac{\partial^3 w_0}{\partial \xi^3}\right. \\
&\left.+3\frac{\partial^3 w_0}{\partial \xi^2 \partial \eta} + 3\frac{\partial^3 w_0}{\partial \xi \partial \eta^2} + \frac{\partial^3 w_0}{\partial \eta^3}\right) - \frac{1}{2}\left(\frac{\partial^2 w_1}{\partial \xi^2} + 2\frac{\partial^2 w_1}{\partial \xi \partial \eta} + \frac{\partial^2 w_1}{\partial \eta^2}\right) \\
&\times \left(\frac{\partial \zeta_0}{\partial \eta} + \frac{\partial \zeta_0}{\partial \xi}\right) - \frac{1}{2}\left(\frac{\partial^2 w_0}{\partial \xi^2} + 2\frac{\partial^2 w_0}{\partial \xi \partial \eta} + \frac{\partial^2 w_0}{\partial \eta^2}\right)\left(\frac{\partial \zeta_1}{\partial \eta} + \frac{\partial \zeta_1}{\partial \xi}\right) \\
&+ \frac{\partial Q_0}{\partial \xi}\frac{\partial}{\partial \eta}(\zeta_1 - w_1) - \frac{\partial P_0}{\partial \eta}\frac{\partial}{\partial \xi}(\zeta_1 + w_1) + \left(7\frac{\partial P_0}{\partial \xi} + 6\frac{\partial Q_0}{\partial \eta}\right. \\
&\left.-6\frac{\partial P_0}{\partial \eta}\frac{\partial Q_0}{\partial \xi}\right)\frac{\partial}{\partial \eta}(\zeta_1 + w_1) - \left(7\frac{\partial Q_0}{\partial \eta} + 6\frac{\partial P_0}{\partial \xi} - 6\frac{\partial P_0}{\partial \eta}\frac{\partial Q_0}{\partial \xi}\right) \\
&\times \frac{\partial}{\partial \xi}(\zeta_1 - w_1) - \frac{dq_1}{d\tau}\frac{\partial \zeta_0}{\partial \eta} - \frac{dp_1}{d\tau}\frac{\partial \zeta_0}{\partial \xi} - \frac{1}{5040}\left(\frac{\partial^7 w_0}{\partial \xi^7} + 7\frac{\partial^7 w_0}{\partial \xi^6 \partial \eta}\right. \\
&+21\frac{\partial^7 w_0}{\partial \xi^5 \partial \eta^2} + 35\frac{\partial^7 w_0}{\partial \xi^4 \partial \eta^3} + 35\frac{\partial^7 w_0}{\partial \xi^3 \partial \eta^4} + 21\frac{\partial^7 w_0}{\partial \xi^2 \partial \eta^5} + 7\frac{\partial^7 w_0}{\partial \xi \partial \eta^6}
\end{aligned}
$$

$$+\frac{\partial^7 w_0}{\partial\eta^7}\bigg) + \frac{\zeta_0}{24}\left(\frac{\partial^5 w_0}{\partial\xi^5} + 5\frac{\partial^5 w_0}{\partial\xi^4\partial\eta} + 10\frac{\partial^5 w_0}{\partial\xi^3\partial\eta^2} + 10\frac{\partial^5 w_0}{\partial\xi^2\partial\eta^3}\right.$$

$$\left.+5\frac{\partial^5 w_0}{\partial\xi\partial\eta^4} + \frac{\partial^5 w_0}{\partial\eta^5}\right) - \frac{\zeta_0^2}{2}\left(\frac{\partial^3 w_0}{\partial\xi^3} + 3\frac{\partial^3 w_0}{\partial\xi^2\partial\eta} + 3\frac{\partial^3 w_0}{\partial\xi\partial\eta^2} + \frac{\partial^3 w_0}{\partial\eta^3}\right)$$

$$+\frac{1}{24}\left(\frac{\partial^4 w_0}{\partial\xi^4} + 4\frac{\partial^4 w_0}{\partial\xi^3\partial\eta} + 6\frac{\partial^4 w_0}{\partial\xi^2\partial\eta^2} + 4\frac{\partial^4 w_0}{\partial\xi\partial\eta^3} + \frac{\partial^4 w_0}{\partial\eta^4}\right)\left(\frac{\partial\zeta_0}{\partial\eta}\right.$$

$$\left.+\frac{\partial\zeta_0}{\partial\xi}\right) - \frac{1}{22}\left(\frac{\partial^2 w_0}{\partial\xi^2} + 2\frac{\partial^2 w_0}{\partial\xi\partial\eta} + \frac{\partial^2 w_0}{\partial\eta^2}\right)\left(\frac{\partial}{\partial\eta}(\zeta_0^2) + \frac{\partial}{\partial\xi}(\zeta_0^2)\right)$$

$$+\frac{\partial Q_1}{\partial\xi}\frac{\partial}{\partial\eta}(\zeta_0 - w_0) - \frac{\partial P_1}{\partial\eta}\frac{\partial}{\partial\xi}(\zeta_0 + w_0) - \frac{\partial Q_0}{\partial\xi}\frac{\partial}{\partial\eta}(\zeta_0 w_0)$$

$$-\frac{\partial P_0}{\partial\eta}\frac{\partial}{\partial\xi}(\zeta_0 w_0) - \frac{\partial Q_0}{\partial\tau}\frac{\partial\zeta_0}{\partial\eta} - \frac{\partial P_0}{\partial\tau}\frac{\partial\zeta_0}{\partial\xi} + \left(7\frac{\partial P_1}{\partial\xi} + 6\frac{\partial Q_1}{\partial\eta}\right.$$

$$\left.-6\frac{\partial P_1}{\partial\eta}\frac{\partial Q_0}{\partial\xi} - 6\frac{\partial P_0}{\partial\eta}\frac{\partial Q_1}{\partial\xi}\right)\frac{\partial}{\partial\eta}(\zeta_0 + w_0) - \left(7\frac{\partial Q_1}{\partial\eta} + 6\frac{\partial P_1}{\partial\xi}\right.$$

$$\left.-6\frac{\partial P_1}{\partial\eta}\frac{\partial Q_0}{\partial\xi} - 6\frac{\partial P_0}{\partial\eta}\frac{\partial Q_1}{\partial\xi}\right)\frac{\partial}{\partial\xi}(\zeta_0 - w_0) + \left(7\frac{\partial P_0}{\partial\xi} + 6\frac{\partial Q_0}{\partial\eta}\right.$$

$$\left.-6\frac{\partial P_0}{\partial\eta}\frac{\partial Q_0}{\partial\xi}\right)\frac{\partial}{\partial\eta}(\zeta_0 w_0) + \left(7\frac{\partial Q_0}{\partial\eta} + 6\frac{\partial P_0}{\partial\xi} - 6\frac{\partial P_0}{\partial\eta}\frac{\partial Q_0}{\partial\xi}\right)\frac{\partial}{\partial\xi}(\zeta_0 w_0)$$

$$+\frac{1}{6}\left(\frac{\partial^3}{\partial\xi^3}(Q_0 - P_0) + 2\frac{\partial^3}{\partial\xi^2\partial\eta}(Q_0 - P_0) + \frac{\partial^3}{\partial\xi\partial\eta^2}(Q_0 - P_0)\right)\frac{\partial w_0}{\partial\eta}$$

$$-\frac{1}{6}\left(\frac{\partial^3}{\partial\eta^3}(Q_0 - P_0) + 2\frac{\partial^3}{\partial\xi\partial\eta^2}(Q_0 - P_0) + \frac{\partial^3}{\partial\xi^2\partial\eta}(Q_0 - P_0)\right)\frac{\partial w_0}{\partial\xi}$$

$$+\frac{1}{2}\left(\frac{\partial^2}{\partial\xi^2}(Q_0 - P_0) + \frac{\partial^2}{\partial\xi\partial\eta}(Q_0 - P_0)\right)\frac{\partial^2 w_0}{\partial\eta^2} - \frac{1}{2}\left(\frac{\partial^2}{\partial\eta^2}(Q_0 - P_0)\right.$$

$$\left.+\frac{\partial^2}{\partial\xi\partial\eta}(Q_0 - P_0)\right)\frac{\partial^2 w_0}{\partial\xi^2} + \frac{1}{2}\left(\frac{\partial^2}{\partial\xi^2}(Q_0 - P_0) - \frac{\partial^2}{\partial\eta^2}(Q_0 - P_0)\right)$$

$$\times\frac{\partial^2 w_0}{\partial\xi\partial\eta} - \left(\frac{7}{6}\frac{\partial P_0}{\partial\xi} + \frac{2}{3}\frac{\partial Q_0}{\partial\eta} - \frac{1}{2}\frac{\partial Q_0}{\partial\xi} - \frac{2}{3}\frac{\partial P_0}{\partial\eta}\frac{\partial Q_0}{\partial\xi}\right)\frac{\partial^3 w_0}{\partial\eta^3}$$

$$-\left(\frac{7}{6}\frac{\partial Q_0}{\partial\eta} + \frac{2}{3}\frac{\partial P_0}{\partial\xi} - \frac{1}{2}\frac{\partial P_0}{\partial\eta} - \frac{2}{3}\frac{\partial P_0}{\partial\eta}\frac{\partial Q_0}{\partial\xi}\right)\frac{\partial^3 w_0}{\partial\xi^3}$$

$$+\left(\frac{\partial P_0}{\partial\eta} - 3\frac{\partial Q_0}{\partial\eta} - \frac{5}{2}\frac{\partial P_0}{\partial\xi} + \frac{1}{2}\frac{\partial Q_0}{\partial\xi} + 2\frac{\partial P_0}{\partial\eta}\frac{\partial Q_0}{\partial\xi}\right)\frac{\partial^3 w_0}{\partial\xi^2\partial\eta}$$

$$+\left(\frac{\partial Q_0}{\partial \xi}-3\frac{\partial P_0}{\partial \xi}-\frac{5}{2}\frac{\partial Q_0}{\partial \eta}+\frac{1}{2}\frac{\partial P_0}{\partial \eta}+2\frac{\partial P_0}{\partial \eta}\frac{\partial Q_0}{\partial \xi}\right)\frac{\partial^3 w_0}{\partial \xi\partial \eta^2}$$

$$+7\left(\frac{\partial P_0}{\partial \xi}+\frac{\partial Q_0}{\partial \eta}-\frac{\partial P_0}{\partial \eta}\frac{\partial Q_0}{\partial \xi}\right)\frac{\partial \zeta_0}{\partial \tau}=0, \qquad \text{(C.1)}$$

$$\frac{\partial \zeta_3}{\partial \eta}+\frac{\partial \zeta_3}{\partial \xi}+\frac{\partial w_3}{\partial \eta}-\frac{\partial w_3}{\partial \xi}+\frac{\partial w_2}{\partial \tau}+\frac{\partial}{\partial \eta}(w_0w_2)+\frac{\partial}{\partial \xi}(w_0w_2)$$

$$+\frac{1}{2}\left(\frac{\partial^3 w_2}{\partial \xi^3}+\frac{\partial^3 w_2}{\partial \xi^2\partial \eta}-\frac{\partial^3 w_2}{\partial \xi\partial \eta^2}-\frac{\partial^3 w_2}{\partial \eta^3}\right)-\frac{dq_0}{d\tau}\frac{\partial w_1}{\partial \eta}-\frac{dp_0}{d\tau}\frac{\partial w_1}{\partial \xi}$$

$$+\frac{1}{2}\frac{\partial}{\partial \eta}(w_1^2)+\frac{1}{2}\frac{\partial}{\partial \xi}(w_1^2)-\frac{1}{24}\left(\frac{\partial^5 w_1}{\partial \xi^5}+3\frac{\partial^5 w_1}{\partial \xi^4\partial \eta}+2\frac{\partial^5 w_1}{\partial \xi^3\partial \eta^2}\right.$$

$$\left.-2\frac{\partial^5 w_1}{\partial \xi^2\partial \eta^3}-3\frac{\partial^5 w_1}{\partial \xi\partial \eta^4}-\frac{\partial^5 w_1}{\partial \eta^5}\right)-\frac{1}{2}\frac{\partial}{\partial \tau}\left(\frac{\partial^2 w_1}{\partial \xi^2}+2\frac{\partial^2 w_1}{\partial \xi\partial \eta}+\frac{\partial^2 w_1}{\partial \eta^2}\right)$$

$$-\frac{w_0}{2}\left(\frac{\partial^3 w_1}{\partial \xi^3}+3\frac{\partial^3 w_1}{\partial \xi^2\partial \eta}+3\frac{\partial^3 w_1}{\partial \xi\partial \eta^2}+\frac{\partial^3 w_1}{\partial \eta^3}\right)+\zeta_0\left(\frac{\partial^3 w_1}{\partial \xi^3}+\frac{\partial^3 w_1}{\partial \xi^2\partial \eta}\right.$$

$$\left.-\frac{\partial^3 w_1}{\partial \xi\partial \eta^2}-\frac{\partial^3 w_1}{\partial \eta^3}\right)+\frac{1}{2}\left(\frac{\partial^2 w_1}{\partial \xi^2}+2\frac{\partial^2 w_1}{\partial \xi\partial \eta}+\frac{\partial^2 w_1}{\partial \eta^2}\right)\left(\frac{\partial w_0}{\partial \eta}+\frac{\partial w_0}{\partial \xi}\right)$$

$$+\left(\frac{\partial^2 w_1}{\partial \xi^2}-\frac{\partial^2 w_1}{\partial \eta^2}\right)\left(\frac{\partial \zeta_0}{\partial \eta}+\frac{\partial \zeta_0}{\partial \xi}\right)+\frac{1}{2}\left(\frac{\partial w_1}{\partial \eta}+\frac{\partial w_1}{\partial \xi}\right)\left(\frac{\partial^2 w_0}{\partial \xi^2}\right.$$

$$\left.+2\frac{\partial^2 w_0}{\partial \xi\partial \eta}+\frac{\partial^2 w_0}{\partial \eta^2}\right)+\left(\frac{\partial \zeta_1}{\partial \eta}+\frac{\partial \zeta_1}{\partial \xi}\right)\left(\frac{\partial^2 w_0}{\partial \xi^2}-\frac{\partial^2 w_0}{\partial \eta^2}\right)-\frac{w_1}{2}\left(\frac{\partial^3 w_0}{\partial \xi^3}\right.$$

$$\left.+3\frac{\partial^3 w_0}{\partial \xi^2\partial \eta}+3\frac{\partial^3 w_0}{\partial \xi\partial \eta^2}+\frac{\partial^3 w_0}{\partial \eta^3}\right)+\zeta_1\left(\frac{\partial^3 w_0}{\partial \xi^3}+\frac{\partial^3 w_0}{\partial \xi^2\partial \eta}-\frac{\partial^3 w_0}{\partial \xi\partial \eta^2}\right.$$

$$\left.-\frac{\partial^3 w_0}{\partial \eta^3}\right)-\frac{\partial Q_0}{\partial \xi}\frac{\partial}{\partial \eta}(\zeta_1-w_1)-\frac{\partial P_0}{\partial \eta}\frac{\partial}{\partial \xi}(\zeta_1+w_1)+\left(7\frac{\partial P_0}{\partial \xi}+6\frac{\partial Q_0}{\partial \eta}\right.$$

$$\left.-6\frac{\partial P_0}{\partial \eta}\frac{\partial Q_0}{\partial \xi}\right)\frac{\partial}{\partial \eta}(\zeta_1+w_1)+\left(7\frac{\partial Q_0}{\partial \eta}+6\frac{\partial P_0}{\partial \xi}-6\frac{\partial P_0}{\partial \eta}\frac{\partial Q_0}{\partial \xi}\right)$$

$$\times\frac{\partial}{\partial \xi}(\zeta_1-w_1)-\frac{dq_1}{d\tau}\frac{\partial w_0}{\partial \eta}-\frac{dp_1}{d\tau}\frac{\partial w_0}{\partial \xi}+\frac{1}{720}\left(\frac{\partial^7 w_0}{\partial \xi^7}+5\frac{\partial^7 w_0}{\partial \xi^6\partial \eta}\right.$$

$$\left.+9\frac{\partial^7 w_0}{\partial \xi^5\partial \eta^2}+5\frac{\partial^7 w_0}{\partial \xi^4\partial \eta^3}-5\frac{\partial^7 w_0}{\partial \xi^3\partial \eta^4}-9\frac{\partial^7 w_0}{\partial \xi^2\partial \eta^5}-5\frac{\partial^7 w_0}{\partial \xi\partial \eta^6}-\frac{\partial^7 w_0}{\partial \eta^7}\right)$$

$$+\frac{1}{24}\frac{\partial}{\partial \tau}\left(\frac{\partial^4 w_0}{\partial \xi^4}+4\frac{\partial^4 w_0}{\partial \xi^3\partial \eta}+6\frac{\partial^4 w_0}{\partial \xi^2\partial \eta^2}+4\frac{\partial^4 w_0}{\partial \xi\partial \eta^3}+\frac{\partial^4 w_0}{\partial \eta^4}\right)$$

$$
\begin{aligned}
&+\frac{w_0}{24}\left(\frac{\partial^5 w_0}{\partial\xi^5}+5\frac{\partial^5 w_0}{\partial\xi^4\partial\eta}+10\frac{\partial^5 w_0}{\partial\xi^3\partial\eta^2}+10\frac{\partial^5 w_0}{\partial\xi^2\partial\eta^3}+5\frac{\partial^5 w_0}{\partial\xi\partial\eta^4}\right.\\
&\left.+\frac{\partial^5 w_0}{\partial\eta^5}\right)-\frac{\zeta_0}{6}\left(\frac{\partial^5 w_0}{\partial\xi^5}+3\frac{\partial^5 w_0}{\partial\xi^4\partial\eta}+2\frac{\partial^5 w_0}{\partial\xi^3\partial\eta^2}-2\frac{\partial^5 w_0}{\partial\xi^2\partial\eta^3}-3\frac{\partial^5 w_0}{\partial\xi\partial\eta^4}\right.\\
&\left.-\frac{\partial^5 w_0}{\partial\eta^5}\right)-\frac{1}{8}\left(\frac{\partial^4 w_0}{\partial\xi^4}+4\frac{\partial^4 w_0}{\partial\xi^3\partial\eta}+6\frac{\partial^4 w_0}{\partial\xi^2\partial\eta^2}+4\frac{\partial^4 w_0}{\partial\xi\partial\eta^3}+\frac{\partial^4 w_0}{\partial\eta^4}\right)\\
&\times\left(\frac{\partial w_0}{\partial\eta}+\frac{\partial w_0}{\partial\xi}\right)-\frac{1}{6}\left(\frac{\partial^4 w_0}{\partial\xi^4}+2\frac{\partial^4 w_0}{\partial\xi^3\partial\eta}-2\frac{\partial^4 w_0}{\partial\xi\partial\eta^3}-\frac{\partial^4 w_0}{\partial\eta^4}\right)\\
&\times\left(\frac{\partial\zeta_0}{\partial\eta}+\frac{\partial\zeta_0}{\partial\xi}\right)+\frac{\zeta_0^2}{2}\left(\frac{\partial^3 w_0}{\partial\xi^3}+\frac{\partial^3 w_0}{\partial\xi^2\partial\eta}-\frac{\partial^3 w_0}{\partial\xi\partial\eta^2}-\frac{\partial^3 w_0}{\partial\eta^3}\right)\\
&+\frac{1}{2}\frac{dq_0}{d\tau}\left(\frac{\partial^3 w_0}{\partial\xi^2\partial\eta}+2\frac{\partial^3 w_0}{\partial\xi\partial\eta^2}+\frac{\partial^3 w_0}{\partial\eta^3}\right)+\frac{1}{2}\frac{dp_0}{d\tau}\left(\frac{\partial^3 w_0}{\partial\xi^3}+2\frac{\partial^3 w_0}{\partial\xi^2\partial\eta}\right.\\
&\left.+\frac{\partial^3 w_0}{\partial\xi\partial\eta^2}\right)-\zeta_0\frac{\partial}{\partial\tau}\left(\frac{\partial^2 w_0}{\partial\xi^2}+2\frac{\partial^2 w_0}{\partial\xi\partial\eta}+\frac{\partial^2 w_0}{\partial\eta^2}\right)-\zeta_0 w_0\left(\frac{\partial^3 w_0}{\partial\xi^3}\right.\\
&\left.+3\frac{\partial^3 w_0}{\partial\xi^2\partial\eta}+3\frac{\partial^3 w_0}{\partial\xi\partial\eta^2}+\frac{\partial^3 w_0}{\partial\eta^3}\right)+\frac{1}{12}\left(\frac{\partial^3 w_0}{\partial\xi^3}+3\frac{\partial^3 w_0}{\partial\xi^2\partial\eta}+3\frac{\partial^3 w_0}{\partial\xi\partial\eta^2}\right.\\
&\left.+\frac{\partial^3 w_0}{\partial\eta^3}\right)\left(\frac{\partial^2 w_0}{\partial\xi^2}+2\frac{\partial^2 w_0}{\partial\xi\partial\eta}+\frac{\partial^2 w_0}{\partial\eta^2}\right)+\left(\frac{\partial^2 w_0}{\partial\xi^2}+2\frac{\partial^2 w_0}{\partial\xi\partial\eta}+\frac{\partial^2 w_0}{\partial\eta^2}\right)\\
&\times\left(\zeta_0\left(\frac{\partial w_0}{\partial\xi}+\frac{\partial w_0}{\partial\eta}\right)-w_0\left(\frac{\partial\zeta_0}{\partial\xi}+\frac{\partial\zeta_0}{\partial\eta}\right)\right)+\frac{1}{2}\left(\frac{\partial^2 w_0}{\partial\xi^2}-\frac{\partial^2 w_0}{\partial\eta^2}\right)\\
&\times\left(\frac{\partial}{\partial\xi}(\zeta_0^2)+\frac{\partial}{\partial\eta}(\zeta_0^2)\right)-\left(\frac{\partial^2 w_0}{\partial\xi\partial\tau}-\frac{\partial^2 w_0}{\partial\eta\partial\tau}\right)\left(\frac{\partial\zeta_0}{\partial\xi}+\frac{\partial\zeta_0}{\partial\eta}\right)\\
&+\left(\left(\frac{\partial w_0}{\partial\xi}\right)^2+2\frac{\partial w_0}{\partial\xi}\frac{\partial w_0}{\partial\eta}+\left(\frac{\partial w_0}{\partial\eta}\right)^2\right)\left(\frac{\partial\zeta_0}{\partial\xi}+\frac{\partial\zeta_0}{\partial\eta}\right)\\
&-\frac{\partial Q_1}{\partial\xi}\frac{\partial}{\partial\eta}(\zeta_0-w_0)-\frac{\partial P_1}{\partial\eta}\frac{\partial}{\partial\xi}(\zeta_0+w_0)-\frac{1}{2}\frac{\partial Q_0}{\partial\xi}\frac{\partial}{\partial\eta}(w_0^2)\\
&-\frac{1}{2}\frac{\partial P_0}{\partial\eta}\frac{\partial}{\partial\xi}(w_0^2)-\frac{\partial Q_0}{\partial\tau}\frac{\partial w_0}{\partial\eta}-\frac{\partial P_0}{\partial\tau}\frac{\partial w_0}{\partial\xi}+\left(7\frac{\partial P_1}{\partial\xi}+6\frac{\partial Q_1}{\partial\eta}\right.\\
&\left.-6\frac{\partial P_1}{\partial\eta}\frac{\partial Q_0}{\partial\xi}-6\frac{\partial P_0}{\partial\eta}\frac{\partial Q_1}{\partial\xi}\right)\frac{\partial}{\partial\eta}(\zeta_0+w_0)+\left(7\frac{\partial Q_1}{\partial\eta}+6\frac{\partial P_1}{\partial\xi}\right.\\
&\left.-6\frac{\partial P_1}{\partial\eta}\frac{\partial Q_0}{\partial\xi}-6\frac{\partial P_0}{\partial\eta}\frac{\partial Q_1}{\partial\xi}\right)\frac{\partial}{\partial\xi}(\zeta_0-w_0)+\frac{1}{2}\left(7\frac{\partial P_0}{\partial\xi}+6\frac{\partial Q_0}{\partial\eta}\right.
\end{aligned}
$$

$$
-6\frac{\partial P_0}{\partial \eta}\frac{\partial Q_0}{\partial \xi}\Big) \frac{\partial}{\partial \eta}(w_0^2) + \frac{1}{2}\left(7\frac{\partial Q_0}{\partial \eta} + 6\frac{\partial P_0}{\partial \xi} - 6\frac{\partial P_0}{\partial \eta}\frac{\partial Q_0}{\partial \xi}\right)\frac{\partial}{\partial \xi}(w_0^2)
$$

$$
-\frac{1}{2}\left(\frac{\partial^3}{\partial \xi^3}(Q_0 - P_0) - \frac{\partial^3}{\partial \xi \partial \eta^2}(Q_0 - P_0)\right)\frac{\partial w_0}{\partial \eta} - \frac{1}{2}\left(\frac{\partial^3}{\partial \eta^3}(Q_0 - P_0)\right.
$$

$$
\left.-\frac{\partial^3}{\partial \xi^2 \partial \eta}(Q_0 - P_0)\right)\frac{\partial w_0}{\partial \xi} - \frac{1}{2}\left(\frac{\partial^2}{\partial \xi^2}(Q_0 + P_0) + \frac{\partial^2}{\partial \xi \partial \eta}(3P_0 - Q_0)\right)
$$

$$
\times \frac{\partial^2 w_0}{\partial \eta^2} + \frac{1}{2}\left(\frac{\partial^2}{\partial \eta^2}(Q_0 + P_0) + \frac{\partial^2}{\partial \xi \partial \eta}(3Q_0 - P_0)\right)\frac{\partial^2 w_0}{\partial \xi^2}
$$

$$
+\frac{1}{2}\left(\frac{\partial^2}{\partial \xi^2}(P_0 - 3Q_0) - \frac{\partial^2}{\partial \eta^2}(Q_0 - 3P_0)\right)\frac{\partial^2 w_0}{\partial \xi \partial \eta}
$$

$$
-\left(\frac{7}{2}\frac{\partial P_0}{\partial \xi} + 2\frac{\partial Q_0}{\partial \eta} - \frac{1}{2}\frac{\partial Q_0}{\partial \xi} - 2\frac{\partial P_0}{\partial \eta}\frac{\partial Q_0}{\partial \xi}\right)\frac{\partial^3 w_0}{\partial \eta^3}
$$

$$
+\left(\frac{7}{2}\frac{\partial Q_0}{\partial \eta} + 2\frac{\partial P_0}{\partial \xi} - \frac{1}{2}\frac{\partial P_0}{\partial \eta} - 2\frac{\partial P_0}{\partial \eta}\frac{\partial Q_0}{\partial \xi}\right)\frac{\partial^3 w_0}{\partial \xi^3}
$$

$$
+\left(\frac{\partial P_0}{\partial \eta} + 3\frac{\partial Q_0}{\partial \eta} + \frac{5}{2}\frac{\partial P_0}{\partial \xi} - \frac{3}{2}\frac{\partial Q_0}{\partial \xi} - 2\frac{\partial P_0}{\partial \eta}\frac{\partial Q_0}{\partial \xi}\right)\frac{\partial^3 w_0}{\partial \xi^2 \partial \eta}
$$

$$
-\left(\frac{\partial Q_0}{\partial \xi} + 3\frac{\partial P_0}{\partial \xi} + \frac{5}{2}\frac{\partial Q_0}{\partial \eta} - \frac{3}{2}\frac{\partial P_0}{\partial \eta} - 2\frac{\partial P_0}{\partial \eta}\frac{\partial Q_0}{\partial \xi}\right)\frac{\partial^3 w_0}{\partial \xi \partial \eta^2}
$$

$$
+7\left(\frac{\partial P_0}{\partial \xi} + \frac{\partial Q_0}{\partial \eta} - \frac{\partial P_0}{\partial \eta}\frac{\partial Q_0}{\partial \xi}\right)\frac{\partial w_0}{\partial \tau} = 0, \tag{C.2}
$$

Appendix D

$\mathcal{O}(\epsilon^4)$ Equations for Chapter 5

$$
\begin{aligned}
&\frac{\partial S_4}{\partial \eta} - \frac{\partial S_4}{\partial \xi} + \frac{\partial u_4}{\partial \eta} + \frac{\partial u_4}{\partial \xi} + \frac{\partial S_3}{\partial \tau} + \frac{\partial}{\partial \eta}(S_1 u_3) + \frac{\partial}{\partial \xi}(S_1 u_3) \\
&+ \frac{\partial}{\partial \eta}(u_1 S_3) + \frac{\partial}{\partial \xi}(u_1 S_3) - \frac{dp_0}{d\tau}\frac{\partial S_2}{\partial \xi} - \frac{dq_0}{d\tau}\frac{\partial S_2}{\partial \eta} + \frac{\partial P_0}{\partial \xi}\frac{\partial}{\partial \eta}(u_2 + S_2) \\
&- \frac{\partial P_0}{\partial \eta}\frac{\partial}{\partial \xi}(u_2 + S_2) - \frac{\partial Q_0}{\partial \xi}\frac{\partial}{\partial \eta}(u_2 - S_2) + \frac{\partial Q_0}{\partial \eta}\frac{\partial}{\partial \xi}(u_2 - S_2) \\
&+ \frac{\partial}{\partial \eta}(S_2 u_2) + \frac{\partial}{\partial \xi}(S_2 u_2) + \frac{\partial P_1}{\partial \xi}\frac{\partial}{\partial \eta}(u_1 + S_1) - \frac{dp_1}{d\tau}\frac{\partial S_1}{\partial \xi} - \frac{dq_1}{d\tau}\frac{\partial S_1}{\partial \eta} \\
&- \frac{\partial P_1}{\partial \eta}\frac{\partial}{\partial \xi}(u_1 + S_1) - \frac{\partial Q_1}{\partial \xi}\frac{\partial}{\partial \eta}(u_1 - S_1) + \frac{\partial Q_1}{\partial \eta}\frac{\partial}{\partial \xi}(u_1 - S_1) \\
&+ \left(\frac{\partial P_0}{\partial \xi} - \frac{\partial Q_0}{\partial \xi}\right)\frac{\partial}{\partial \eta}(u_1 S_1) - \left(\frac{\partial P_0}{\partial \eta} - \frac{\partial Q_0}{\partial \eta}\right)\frac{\partial}{\partial \xi}(u_1 S_1) - \frac{\partial P_0}{\partial \tau}\frac{\partial S_1}{\partial \xi} \\
&- \frac{\partial Q_0}{\partial \tau}\frac{\partial S_1}{\partial \eta} + \left(\frac{\partial P_0}{\partial \xi} + \frac{\partial Q_0}{\partial \eta} - \frac{\partial P_0}{\partial \eta}\frac{\partial Q_0}{\partial \xi}\right)\frac{\partial S_1}{\partial \tau} = 0,
\end{aligned} \tag{D.1}
$$

$$
\begin{aligned}
&\frac{\partial \pi_4}{\partial \eta} + \frac{\partial \pi_4}{\partial \xi} + \frac{\partial u_4}{\partial \eta} - \frac{\partial u_4}{\partial \xi} + \frac{\partial u_3}{\partial \tau} + \frac{\partial}{\partial \eta}(u_1 u_3) + \frac{\partial}{\partial \xi}(u_1 u_3) \\
&+ \frac{1}{2}\frac{\partial}{\partial \eta}(u_2^2) + \frac{1}{2}\frac{\partial}{\partial \xi}(u_2^2) - \frac{dp_0}{d\tau}\frac{\partial u_2}{\partial \xi} - \frac{dq_0}{d\tau}\frac{\partial u_2}{\partial \eta} + \frac{\partial P_0}{\partial \xi}\frac{\partial}{\partial \eta}(u_2 + \pi_2) \\
&- \frac{\partial P_0}{\partial \eta}\frac{\partial}{\partial \xi}(u_2 + \pi_2) + \frac{\partial Q_0}{\partial \xi}\frac{\partial}{\partial \eta}(u_2 - \pi_2) - \frac{\partial Q_0}{\partial \eta}\frac{\partial}{\partial \xi}(u_2 - \pi_2)
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{dp_1}{d\tau}\frac{\partial u_1}{\partial \xi}-\frac{dq_1}{d\tau}\frac{\partial u_1}{\partial \eta}+\frac{\partial P_1}{\partial \xi}\frac{\partial}{\partial \eta}(u_1+\pi_1)-\frac{\partial P_1}{\partial \eta}\frac{\partial}{\partial \xi}(u_1+\pi_1)\\
&+\frac{\partial Q_1}{\partial \xi}\frac{\partial}{\partial \eta}(u_1-\pi_1)-\frac{\partial Q_1}{\partial \eta}\frac{\partial}{\partial \xi}(u_1-\pi_1)+\frac{1}{2}\left(\frac{\partial P_0}{\partial \xi}-\frac{\partial Q_0}{\partial \xi}\right)\frac{\partial}{\partial \eta}(u_1^2)\\
&-\frac{1}{2}\left(\frac{\partial P_0}{\partial \eta}-\frac{\partial Q_0}{\partial \eta}\right)\frac{\partial}{\partial \xi}(u_1^2)-\frac{\partial P_0}{\partial \tau}\frac{\partial u_1}{\partial \xi}-\frac{\partial Q_0}{\partial \tau}\frac{\partial u_1}{\partial \eta}\\
&+\left(\frac{\partial P_0}{\partial \xi}+\frac{\partial Q_0}{\partial \eta}-\frac{\partial P_0}{\partial \eta}\frac{\partial Q_0}{\partial \xi}\right)\frac{\partial u_1}{\partial \tau}=0,
\end{aligned}
\tag{D.2}
$$

$$
\begin{aligned}
\pi_4 =&S_4+\alpha S_1S_3-\pi_1S_3-S_1\pi_3+\frac{\partial^2 S_3}{\partial \eta^2}+\frac{\partial^2 S_3}{\partial \xi^2}-2\frac{\partial^2 S_3}{\partial \xi\partial \eta}-\pi_2S_2\\
&+\frac{1}{2}\alpha S_2^2-\frac{1}{2}\pi_1S_1S_2-\frac{1}{4}S_1^2\pi_2+\frac{S_1}{2}\left(\frac{\partial^2 S_2}{\partial \eta^2}+\frac{\partial^2 S_2}{\partial \xi^2}-2\frac{\partial^2 S_2}{\partial \xi\partial \eta}\right)\\
&+\frac{S_2}{2}\left(\frac{\partial^2 S_1}{\partial \eta^2}+\frac{\partial^2 S_1}{\partial \xi^2}-2\frac{\partial^2 S_1}{\partial \xi\partial \eta}\right)+2\frac{\partial}{\partial \tau}\left(\frac{\partial S_2}{\partial \eta}-\frac{\partial S_2}{\partial \xi}\right)\\
&-2\left(\frac{\partial P_0}{\partial \xi}+\frac{\partial Q_0}{\partial \eta}-\frac{\partial P_0}{\partial \eta}\frac{\partial Q_0}{\partial \xi}\right)(\pi_2-S_2+\pi_1S_1)+\frac{\partial^2 S_1}{\partial \tau^2}\\
&+\frac{\partial}{\partial \tau}\left(\frac{\partial S_1}{\partial \eta}-\frac{\partial S_1}{\partial \xi}\right)S_1+2\frac{dp_0}{d\tau}\frac{\partial^2 S_1}{\partial \xi^2}-2\frac{dq_0}{d\tau}\frac{\partial^2 S_1}{\partial \eta^2}+2\left(\frac{dq_0}{d\tau}\right.\\
&\left.-\frac{dp_0}{d\tau}\right)\frac{\partial^2 S_1}{\partial \xi\partial \eta}+2\left(\frac{\partial P_1}{\partial \xi}+\frac{\partial Q_1}{\partial \eta}-\frac{\partial P_1}{\partial \eta}\frac{\partial Q_0}{\partial \xi}-\frac{\partial P_0}{\partial \eta}\frac{\partial Q_1}{\partial \xi}\right)(S_1\\
&-\pi_1)+2\frac{\partial}{\partial \xi}(P_0+Q_0)\frac{\partial^2 S_1}{\partial \eta^2}+2\frac{\partial}{\partial \eta}(P_0+Q_0)\frac{\partial^2 S_1}{\partial \xi^2}\\
&-2\left(\frac{\partial}{\partial \xi}(P_0+Q_0)+\frac{\partial}{\partial \eta}(P_0+Q_0)\right)\frac{\partial^2 S_1}{\partial \xi\partial \eta}-\frac{\partial^2}{\partial \xi^2}(P_0+Q_0)\frac{\partial S_1}{\partial \eta}\\
&-\frac{\partial^2}{\partial \eta^2}(P_0+Q_0)\frac{\partial S_1}{\partial \xi}+\left(\frac{\partial^2 P_0}{\partial \xi\partial \eta}+\frac{\partial^2 Q_0}{\partial \xi\partial \eta}\right)\left(\frac{\partial S_1}{\partial \eta}+\frac{\partial S_1}{\partial \xi}\right).
\end{aligned}
\tag{D.3}
$$

Curriculum Vitae

Publications:

[1] A.E. Ozden and H. Demiray, Re-visiting the head-on collision problem between two solitary waves in shallow water, *International Journal of Non-Linear Mechanics*, **69**, 66-70, 2015.

[2] A.E. Ozden and H. Demiray, On head-on collision between two solitary waves in shallow water: the use of the extended PLK method, *Nonlinear Dynamics*, **82**, 73-84, 2015.