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On spherical submanifolds with finite type spherical Gauss map

Abstract: Chen and Lue (2007) initiated the study of spherical submanifolds with finite type spherical Gauss map. In this paper, we firstly prove that a submanifold M^n of the unit sphere \mathbb{S}^{m-1} has non-mass-symmetric 1-type spherical Gauss map if and only if M^n is an open part of a small n -sphere of a totally geodesic $(n+1)$ -sphere $\mathbb{S}^{n+1} \subset \mathbb{S}^{m-1}$. Then we show that a non-totally umbilical hypersurface M of \mathbb{S}^{n+1} with nonzero constant mean curvature in \mathbb{S}^{n+1} has mass-symmetric 2-type spherical Gauss map if and only if the scalar curvature of M is constant. Finally, we classify constant mean curvature surfaces in \mathbb{S}^3 with mass-symmetric 2-type spherical Gauss map.

Keywords: Finite type map, spherical Gauss map, mean curvature, isoparametric hypersurface.

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1 Introduction

Let M^n denote a Riemannian n -manifold with Laplacian operator Δ . A smooth map $\varphi : M^n \rightarrow \mathbb{E}^N$ of M^n into the Euclidean N -space is said to be of *finite type* if it admits a finite spectral decomposition:

$$\varphi = c + \sum_{t=1}^k \varphi_t, \quad (1)$$

where c is a constant vector in \mathbb{E}^N , each φ_t is a non-constant \mathbb{E}^N -valued maps satisfying $\Delta \varphi_t = \lambda_{p_t} \varphi_t$ with $\lambda_{p_t} \in \mathbb{R}$ and $\lambda_{p_1} < \lambda_{p_2} < \dots < \lambda_{p_k}$. Otherwise, φ is said to be of infinite type. When the spectral resolution (1) contains exactly k non-constant terms, the map φ is called of *k-type* (see [3; 4] for details).

Let $\mathbb{S}^{N-1}(x_0, c_0) \subset \mathbb{E}^N$ denote a hypersphere of \mathbb{E}^N with curvature $c_0 > 0$, where $x_0 \in \mathbb{E}^N$ is the center of the sphere. If x_0 is the origin of \mathbb{E}^N and $c_0 = 1$, we denote the unit hypersphere $\mathbb{S}^{N-1}(0, 1)$ by \mathbb{S}^{N-1} .

A spherical finite type map $\varphi : M^n \rightarrow \mathbb{S}^{N-1} \subset \mathbb{E}^N$ of a Riemannian manifold M into \mathbb{S}^{N-1} is called *mass-symmetric* if the vector c in its spectral resolution is the center of \mathbb{S}^{N-1} (which is the origin of \mathbb{E}^N). Otherwise, φ is called non-mass-symmetric.

Let $\mathbf{x} : M^n \rightarrow \mathbb{E}^m$ be an isometric immersion from a Riemannian n -manifold M^n into a Euclidean m -space \mathbb{E}^m . Let $G(n, m)$ denote the Grassmannian manifold consisting of linear n -subspaces of \mathbb{E}^m . The classical Gauss map $\nu^c : M^n \rightarrow G(n, m)$ associated with \mathbf{x} is the map which carries each point $p \in M$ to the linear subspace of \mathbb{E}^m obtained by parallel displacement of the tangent space $T_p M$ to the origin of \mathbb{E}^m . Since $G(n, m)$ can be canonically imbedded in the vector space $\wedge^n \mathbb{E}^m = \mathbb{E}^N$ with $N = \binom{m}{n}$, obtained by the exterior products of n -vectors in \mathbb{E}^m , the classical Gauss map gives rise to a well-defined map from M^n into the Euclidean N -space \mathbb{E}^N .

In [7], Chen and Piccinni initiated the study of Euclidean submanifolds with finite type classical Gauss map. Since then many geometers have studied such submanifolds, see [2; 1; 5; 8; 9].

For a spherical submanifold M^n in \mathbb{S}^{m-1} , Obata [10] studied the generalized Gauss map which assigns to each $p \in M$ the totally geodesic n -sphere of \mathbb{S}^{m-1} determined by the tangent space $T_p M^n$. Since a totally

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geodesic n -sphere \mathbb{S}^n of \mathbb{S}^{m-1} determines a unique linear $(n + 1)$ -space containing the totally geodesic \mathbb{S}^n in \mathbb{E}^m , Obata’s map can be extended to a map $\hat{\nu}$ of M^n into the Grassmannian $G(n + 1, m)$ in a natural way, known as the spherical Gauss map. The composition $\tilde{\nu}$ of $\hat{\nu}$ followed by the natural inclusion of $G(n + 1, m)$ in $\mathbb{E}^{\binom{m}{n+1}}$ is also called the spherical Gauss map.

Let $\mathbf{x} : M^n \rightarrow \mathbb{S}^{m-1}$ be an isometric immersion of an orientable Riemannian n -manifold into the unit sphere \mathbb{S}^{m-1} . We identify each point p with $\mathbf{x}(p)$ and tangent vector $v \in T_pM$ with its image $d\mathbf{x}_p(v)$ under the differential $d\mathbf{x}_p$. For each point $p \in M^n$, let e_1, \dots, e_n be an oriented orthonormal basis of T_pM^n . Since the $n + 1$ vectors $\mathbf{x}, e_1, \dots, e_n$ determine a linear $(n + 1)$ -subspace in \mathbb{E}^m , the intersection of this linear subspace with \mathbb{S}^{m-1} is a totally geodesic n -sphere determined by T_pM^n as in [10]. Thus the spherical Gauss map $\tilde{\nu} : M^n \rightarrow \mathbb{E}^{\binom{m}{n+1}}$ associated with \mathbf{x} is given by (see [6] for details)

$$\tilde{\nu} = \mathbf{x} \wedge e_1 \wedge \dots \wedge e_n : M^n \rightarrow G(n + 1, m) \subset \mathbb{S}^{\binom{m}{n+1}-1} \subset \mathbb{E}^{\binom{m}{n+1}}. \tag{2}$$

In [6], Chen and Lue studied spherical submanifolds with finite type spherical Gauss map. As they explained the geometric behavior of classical and spherical Gauss map are different. For example, the classical Gauss map of every compact Euclidean submanifold is mass-symmetric; but the spherical Gauss map of a spherical compact submanifold is not mass-symmetric in general. Moreover, by [7] the classical Gauss map of the surface $\mathbb{S}^1(a) \times \mathbb{S}^1(b) \subset \mathbb{S}^3(1) \subset \mathbb{E}^4$, $a^2 + b^2 = 1$, is of 1-type; however we show in this paper that the spherical Gauss map of the surface $\mathbb{S}^1(a) \times \mathbb{S}^1(b) \subset \mathbb{S}^3(1)$ with $a \neq b$ and $a^2 + b^2 = 1$ is mass-symmetric and of 2-type.

In [6], Chen and Lue classified spherical submanifolds with 1-type spherical Gauss map. They also classified minimal surfaces in \mathbb{S}^4 with mass-symmetric 2-type spherical Gauss map, and minimal surfaces in \mathbb{S}^5 with non-mass-symmetric 2-type spherical Gauss map. They stated that every isoparametric hypersurface of \mathbb{S}^{n+1} has 1-type spherical Gauss map. However, the results given for non-mass-symmetric 1-type spherical Gauss map (Theorem 4.3 in [6]) is not true. In this paper, we prove that an n -dimensional submanifold M of \mathbb{S}^{m-1} has non-mass-symmetric 1-type spherical Gauss map if and only if M is an open part of a small n -sphere of a totally geodesic $(n + 1)$ -sphere $\mathbb{S}^{n+1} \subset \mathbb{S}^{m-1}$. We also prove that a non-totally umbilical hypersurface M of \mathbb{S}^{n+1} with nonzero constant mean curvature in \mathbb{S}^{n+1} has mass-symmetric 2-type spherical Gauss map if and only if the scalar curvature of M is constant. Moreover we show that the spherical Gauss map of a non-totally umbilical surface M of \mathbb{S}^3 with nonzero constant mean curvature is mass-symmetric and of 2-type if and only if M is an open part of the surface $\mathbb{S}^1(a) \times \mathbb{S}^1(b) \subset \mathbb{S}^3(1)$ with $a \neq b$ and $a^2 + b^2 = 1$.

2 Preliminaries

Let M be an n -dimensional isometrically immersed submanifold in a Riemannian m -manifold \tilde{M} . Let $\tilde{\nabla}$ be the Levi-Civita connection of \tilde{M} and ∇ the induced connection on M . We choose a local field of orthonormal frame $e_1, \dots, e_n, e_{n+1}, \dots, e_m$ such that, restricted to M , the vectors e_1, \dots, e_n are tangent to M and hence e_{n+1}, \dots, e_m are normal to M . We use the following convention on the range of indices:

$$1 \leq A, B, C, \dots \leq m, \quad 1 \leq i, j, k, \dots \leq n, \quad n + 1 \leq r, s, t, \dots \leq m.$$

With respect to the chosen frame field of M , let $\{\omega^1, \dots, \omega^m\}$ be the field of dual frame and let $\{\omega_{AB}\}$ with $\omega_{AB} + \omega_{BA} = 0$ be the connection forms. Then we have the formulas of Gauss and Weingarten, respectively, as

$$\tilde{\nabla}_{e_k} e_i = \sum_{j=1}^n \omega_{ij}(e_k) e_j + \sum_{r=n+1}^m h_{ik}^r e_r \quad \text{and} \quad \tilde{\nabla}_{e_k} e_r = -A_r(e_k) + \sum_{s=n+1}^m \omega_{rs}(e_k) e_s,$$

where the h_{ij}^r ’s are the coefficients of the second fundamental form h , A_r is the Weingarten map in direction e_r , and ω_{rs} are the normal connection forms. Also, the normal connection is defined by $D_{e_i} e_r = \sum_{s=n+1}^m \omega_{rs}(e_i) e_s$.

The mean curvature vector H and the squared length $\|h\|^2$ of the second fundamental form h are defined, respectively, by

$$H = \frac{1}{n} \sum_{r=n+1}^m \text{tr} A_r e_r \quad \text{and} \quad \|h\|^2 = \sum_{r=n+1}^m \text{tr}(A_r)^2.$$

The Codazzi equation of M is given by

$$h^r_{ij,k} = h^r_{jk,i}, \quad h^r_{jk,i} = e_i(h^r_{jk}) - \sum_{\ell=1}^n (h^r_{j\ell} \omega_{k\ell}(e_i) + h^r_{k\ell} \omega_{j\ell}(e_i)) + \sum_{s=n+1}^m h^s_{jk} \omega_{sr}(e_i). \tag{3}$$

Also, from the Ricci equation of M we have

$$R^D(e_j, e_k; e_r, e_s) = \langle [A_{e_r}, A_{e_s}](e_j), e_k \rangle = \sum_{i=1}^n (h^r_{ki} h^s_{ij} - h^r_{ji} h^s_{ik}), \tag{4}$$

where R^D is the normal curvature tensor.

If the ambient space \widetilde{M} is the Euclidean m -space \mathbb{E}^m , then the scalar curvature S of M is given by

$$S = n^2 |H|^2 - \|h\|^2, \tag{5}$$

where $|H|^2$ is the squared length of the mean curvature vector H of M in \mathbb{E}^m . In particular, if M is immersed in the unit sphere $\mathbb{S}^{m-1} \subset \mathbb{E}^m$, then (5) gives

$$S = n(n-1) + n^2 |\hat{H}|^2 - \|\hat{h}\|^2, \tag{6}$$

where \hat{H} and \hat{h} are the mean curvature vector and the second fundamental form of M in \mathbb{S}^{m-1} , respectively. For M in $\mathbb{S}^{m-1} \subset \mathbb{E}^m$ we also have

$$H = \hat{H} - \mathbf{x}, \quad h(X, Y) = \hat{h}(X, Y) - \langle X, Y \rangle \mathbf{x}. \tag{7}$$

A hypersurface M in \mathbb{S}^{n+1} is said to be *isoparametric* if it has constant principal curvatures.

3 Finite type spherical Gauss map

In [6], the Laplacian of the spherical Gauss map $\tilde{\nu}$ is given by

$$\begin{aligned} \Delta \tilde{\nu} &= \|\hat{h}\|^2 \tilde{\nu} + n \hat{H} \wedge e_1 \wedge \dots \wedge e_n - n \sum_{k=1}^n \mathbf{x} \wedge e_1 \wedge \dots \wedge \underbrace{D_{e_k} \hat{H}}_{k\text{-th}} \wedge \dots \wedge e_n \\ &+ \sum_{j,k;s < r} R^r_{sjk} \mathbf{x} \wedge e_1 \wedge \dots \wedge \underbrace{e_s}_{k\text{-th}} \wedge \dots \wedge \underbrace{e_r}_{j\text{-th}} \wedge \dots \wedge e_n, \end{aligned} \tag{8}$$

where $R^r_{sjk} = R^D(e_j, e_k; e_r, e_s)$. The following two theorems were stated for the submanifolds in \mathbb{S}^{m-1} with 1-type spherical Gauss map.

Theorem 3.1 ([6]). *A submanifold of \mathbb{S}^{m-1} has mass-symmetric 1-type spherical Gauss map if and only if it is a minimal submanifold of \mathbb{S}^{m-1} with constant scalar curvature and flat normal connection.*

Theorem 3.2 ([6]). *An n -dimensional submanifold of \mathbb{S}^{m-1} has non-mass-symmetric 1-type spherical Gauss map if and only if it has constant scalar curvature and it is immersed in a totally geodesic $(n+1)$ -sphere $\mathbb{S}^{n+1} \subset \mathbb{S}^{m-1}$ as a hypersurface with nonzero constant mean curvature.*

By Theorem 3.2, every non-minimal isoparametric hypersurface in \mathbb{S}^{n+1} must have non-mass-symmetric 1-type Gauss map. However, we prove that every non-minimal and non-totally umbilical isoparametric hypersurface in \mathbb{S}^{n+1} has mass-symmetric 2-type spherical Gauss map (see Corollary 3.7). In the proof of Theorem 3.2, Equation (4.2) in [6, p. 414] is incorrect because of two missing terms in that equation. We prove the next theorem for submanifolds in \mathbb{S}^{m-1} with non-mass-symmetric 1-type spherical Gauss map. Also, the statement of Corollary 4.1 in [6] must be as follows:

Corollary 3.3. *Every isoparametric minimal hypersurface of \mathbb{S}^{n+1} has mass-symmetric 1-type spherical Gauss map.*

For later use we prove the following lemma.

Lemma 3.4. For a hypersurface M of $\mathbb{S}^{n+1} \subset \mathbb{E}^{n+2}$ we have

$$\Delta(e_{n+1} \wedge e_1 \wedge e_2 \wedge \cdots \wedge e_n) = n\hat{\alpha}\tilde{\nu} + ne_{n+1} \wedge e_1 \wedge e_2 \wedge \cdots \wedge e_n, \tag{9}$$

where $\hat{\alpha}$ is the mean curvature of M in \mathbb{S}^{n+1} .

Proof. Let $e_1, \dots, e_{n+1}, e_{n+2}$ be a local orthonormal frame field on M in \mathbb{E}^{n+2} such that e_1, e_2, \dots, e_n are tangent to M and $e_{n+1}, e_{n+2} = \mathbf{x}$ are normal to M , where \mathbf{x} is the position vector of M . Since $e_{n+2} = \mathbf{x}$ is parallel in the normal bundle of M in \mathbb{E}^{n+2} and the codimension of M in \mathbb{E}^{n+2} is two, e_{n+1} is parallel too. Let us put $\tilde{\nu} = e_{n+1} \wedge e_1 \wedge e_2 \wedge \cdots \wedge e_n$. Now we will compute $\Delta\tilde{\nu}$. By differentiating $\tilde{\nu}$ we get

$$e_i\tilde{\nu} = -e_{n+1} \wedge e_1 \wedge \cdots \wedge e_{i-1} \wedge \mathbf{x} \wedge e_{i+1} \wedge \cdots \wedge e_n. \tag{10}$$

Since $\nabla_{e_i}e_i = \sum_{j=1}^n \omega_{ij}(e_i)e_j$ and $De_{n+1} = 0$, we have

$$(\nabla_{e_i}e_i)\tilde{\nu} = -\sum_{j=1}^n \omega_{ij}(e_i)e_{n+1} \wedge e_1 \wedge \cdots \wedge e_{j-1} \wedge \mathbf{x} \wedge e_{j+1} \wedge \cdots \wedge e_n. \tag{11}$$

Differentiating $e_i(\tilde{\nu})$ in (10) we obtain that

$$\begin{aligned} e_i e_i \tilde{\nu} &= -\tilde{\nu} - h_{ii}^{n+1} \tilde{\nu} - \sum_{j,\ell=1}^n \omega_{j\ell}(e_i) e_{n+1} \wedge e_1 \wedge \cdots \wedge \underbrace{e_\ell}_{j\text{-th}} \wedge \cdots \wedge \underbrace{\mathbf{x}}_{i\text{-th}} \wedge \cdots \wedge e_n \\ &= -\tilde{\nu} - h_{ii}^{n+1} \tilde{\nu} + \sum_{j=1}^n \omega_{ji}(e_i) e_{n+1} \wedge e_1 \wedge \cdots \wedge \underbrace{\mathbf{x}}_{j\text{-th}} \wedge \cdots \wedge e_n. \end{aligned} \tag{12}$$

Considering $n\hat{\alpha} = \sum_{i=1}^n h_{ii}^{n+1}$ we have

$$\Delta\tilde{\nu} = \sum_{i=1}^n (\nabla_{e_i}e_i - e_i e_i)\tilde{\nu} = n\hat{\alpha}\tilde{\nu} + n\tilde{\nu} - \sum_{i,j=1}^n (\omega_{ij}(e_i) + \omega_{ji}(e_i)) e_{n+1} \wedge e_1 \wedge \cdots \wedge \underbrace{\mathbf{x}}_{j\text{-th}} \wedge \cdots \wedge e_n \tag{13}$$

which gives (9) as $\omega_{ij} + \omega_{ji} = 0$. □

Theorem 3.5. An n -dimensional submanifold M of \mathbb{S}^{m-1} has non-mass-symmetric 1-type spherical Gauss map if and only if M is an open part of a small n -sphere of a totally geodesic $(n + 1)$ -sphere $\mathbb{S}^{n+1} \subset \mathbb{S}^{m-1}$.

Proof. Let $\mathbf{x} : M \rightarrow \mathbb{S}^{m-1}$ be an isometric immersion of a Riemannian n -manifold M into \mathbb{S}^{m-1} . If the spherical Gauss map $\tilde{\nu}$ of \mathbf{x} is non-mass-symmetric 1-type, then we have $\Delta\tilde{\nu} = \lambda_p(\tilde{\nu} - c)$ for some vector c and some real number λ_p . Thus we have

$$(\Delta\tilde{\nu})_i = \lambda_p(\tilde{\nu})_i, \tag{14}$$

where $(\cdot)_i = e_i(\cdot)$. By differentiating $\tilde{\nu}$ in (2) we find

$$e_i\tilde{\nu} = \sum_{r,k} h_{ik}^r \mathbf{x} \wedge e_1 \wedge \cdots \wedge e_{k-1} \wedge e_r \wedge e_{k+1} \wedge \cdots \wedge e_n. \tag{15}$$

On the other hand, by a direct long computation, we obtain from (8) that

$$\begin{aligned} e_i(\Delta\tilde{\nu}) &= (\|\hat{h}\|_i^2)\tilde{\nu} + \|\hat{h}\|^2 \sum_{r=n+1}^{m-1} \sum_{k=1}^n h_{ik}^r \mathbf{x} \wedge e_1 \wedge \cdots \wedge e_{k-1} \wedge e_r \wedge e_{k+1} \wedge \cdots \wedge e_n + 2nD_{e_i}\hat{H} \wedge e_1 \wedge \cdots \wedge e_n \\ &\quad + n \sum_{k=1}^n \sum_{r=n+1}^{m-1} h_{ik}^r \hat{H} \wedge e_1 \wedge \cdots \wedge \underbrace{e_r}_{k\text{-th}} \wedge \cdots \wedge e_n \\ &\quad - n \sum_{k=1}^n \delta_{ik} \hat{H} \wedge e_1 \wedge \cdots \wedge e_{k-1} \wedge \mathbf{x} \wedge e_{k+1} \wedge \cdots \wedge e_n \end{aligned}$$

$$\begin{aligned}
 & -n \sum_{\substack{j,k,l=1 \\ j \neq k}}^n \omega_{jl}(e_i) \mathbf{x} \wedge e_1 \wedge \cdots \wedge \underline{e_l} \wedge \cdots \wedge \underline{D_{e_k} \hat{H}} \wedge \cdots \wedge e_n \\
 & -n \sum_{\substack{j,k=1 \\ j \neq k}}^n \sum_{r=n+1}^{m-1} h_{ij}^r \mathbf{x} \wedge e_1 \wedge \cdots \wedge \underline{e_r} \wedge \cdots \wedge \underline{D_{e_k} \hat{H}} \wedge \cdots \wedge e_n \\
 & +n \sum_{k=1}^n \langle A_{D_{e_k} \hat{H}}(e_i), e_k \rangle \mathbf{x} \wedge e_1 \wedge \cdots \wedge e_n \\
 & -n \sum_{k=1}^n \mathbf{x} \wedge e_1 \wedge \cdots \wedge \underline{D_{e_i} D_{e_k} \hat{H}} \wedge \cdots \wedge e_n \\
 & + \sum_{\substack{r,s=n+1 \\ s < r}}^{m-1} \sum_{\substack{j,k=1 \\ j \neq k}}^n \{e_i(R_{sjk}^r) \mathbf{x} + R_{sjk}^r e_i\} \wedge e_1 \wedge \cdots \wedge \underline{e_s} \wedge \cdots \wedge \underline{e_r} \wedge \cdots \wedge e_n \\
 & + \sum_{\substack{r,s=n+1 \\ s < r}}^{m-1} \sum_{\substack{j,k,l \\ j,k,l \neq}}^n R_{sjk}^r \left\{ \sum_{h=1}^n \omega_{lh}(e_i) \mathbf{x} \wedge e_1 \wedge \cdots \wedge \underline{e_h} \wedge \cdots \wedge \underline{e_s} \wedge \cdots \wedge \underline{e_r} \wedge \cdots \wedge e_n \right. \\
 & \left. + \sum_{t=n+1}^{m-1} h_{il}^t \mathbf{x} \wedge e_1 \wedge \cdots \wedge \underline{e_t} \wedge \cdots \wedge \underline{e_s} \wedge \cdots \wedge \underline{e_r} \wedge \cdots \wedge e_n \right\} \\
 & - \sum_{r,s=n+1}^{m-1} \sum_{\substack{j,k,l=1 \\ j \neq k}}^n R_{sjk}^r h_{il}^s \mathbf{x} \wedge e_1 \wedge \cdots \wedge \underline{e_l} \wedge \cdots \wedge \underline{e_r} \wedge \cdots \wedge e_n \\
 & + \sum_{r,s,t=n+1}^{m-1} \sum_{\substack{j,k=1 \\ j \neq k}}^n R_{sjk}^r \omega_{st}(e_i) \mathbf{x} \wedge e_1 \wedge \cdots \wedge \underline{e_t} \wedge \cdots \wedge \underline{e_r} \wedge \cdots \wedge e_n.
 \end{aligned} \tag{16}$$

Case (a): $\hat{H} = 0$. In this case, equation (16) reduces to

$$\begin{aligned}
 e_i(\Delta \tilde{\nu}) &= (\|\hat{h}\|^2)_i \tilde{\nu} + \|\hat{h}\|^2 \sum_{r=n+1}^{m-1} \sum_{k=1}^n h_{ik}^r \mathbf{x} \wedge e_1 \wedge \cdots \wedge \underline{e_r} \wedge \cdots \wedge e_n \\
 & + \sum_{\substack{r,s=n+1 \\ s < r}}^{m-1} \sum_{\substack{j,k=1 \\ j \neq k}}^n \{e_i(R_{sjk}^r) \mathbf{x} + R_{sjk}^r e_i\} \wedge e_1 \wedge \cdots \wedge \underline{e_s} \wedge \cdots \wedge \underline{e_r} \wedge \cdots \wedge e_n \\
 & + \sum_{\substack{r,s=n+1 \\ s < r}}^{m-1} \sum_{\substack{j,k,l \\ j,k,l \neq}}^n R_{sjk}^r \left\{ \sum_{h=1}^n \omega_{lh}(e_i) \mathbf{x} \wedge e_1 \wedge \cdots \wedge \underline{e_h} \wedge \cdots \wedge \underline{e_s} \wedge \cdots \wedge \underline{e_r} \wedge \cdots \wedge e_n \right. \\
 & \left. + \sum_{t=n+1}^{m-1} h_{il}^t \mathbf{x} \wedge e_1 \wedge \cdots \wedge \underline{e_t} \wedge \cdots \wedge \underline{e_s} \wedge \cdots \wedge \underline{e_r} \wedge \cdots \wedge e_n \right\} \\
 & - \sum_{r,s=n+1}^{m-1} \sum_{\substack{j,k,l=1 \\ j \neq k}}^n R_{sjk}^r h_{il}^s \mathbf{x} \wedge e_1 \wedge \cdots \wedge \underline{e_l} \wedge \cdots \wedge \underline{e_r} \wedge \cdots \wedge e_n \\
 & + \sum_{r,s,t=n+1}^{m-1} \sum_{\substack{j,k=1 \\ j \neq k}}^n R_{sjk}^r \omega_{st}(e_i) \mathbf{x} \wedge e_1 \wedge \cdots \wedge \underline{e_t} \wedge \cdots \wedge \underline{e_r} \wedge \cdots \wedge e_n.
 \end{aligned} \tag{17}$$

Comparing (14), (15) and (17) we get $\|\hat{h}\|_i^2 = R_{sjk}^r = 0$. Thus M has constant scalar curvature and flat normal connection. Theorem 3.1 implies that $\tilde{\nu}$ is mass-symmetric 1-type. This is a contradiction.

Case (b): $\hat{H} \neq 0$. Since the term $D_{e_i} \hat{H} \wedge e_1 \wedge \cdots \wedge e_n$ appears only in $e_i(\Delta \tilde{\nu})$ of (16), but not in $e_i(\tilde{\nu})$, we know from (14), (15) and (16) that $D\hat{H} = 0$. Thus, M has parallel nonzero mean curvature vector in S^{m-1} . So, it has nonzero constant mean curvature. Therefore, equation (16) reduces to

$$e_i(\Delta \tilde{\nu}) = (\|\hat{h}\|^2)_i \tilde{\nu} + \|\hat{h}\|^2 \sum_{r=n+1}^{m-1} \sum_{k=1}^n h_{ik}^r \mathbf{x} \wedge e_1 \wedge \cdots \wedge \underline{e_r} \wedge \cdots \wedge e_n$$

$$\begin{aligned}
 & + n \sum_{k=1}^n \sum_{r=n+1}^{m-1} h_{ik}^r \hat{H} \wedge e_1 \wedge \cdots \wedge e_{k-1} \wedge e_r \wedge e_{k+1} \wedge \cdots \wedge e_n \\
 & - n \sum_{k=1}^n \delta_{ik} \hat{H} \wedge e_1 \wedge \cdots \wedge e_{k-1} \wedge \mathbf{x} \wedge e_{k+1} \wedge \cdots \wedge e_n \\
 & + \sum_{\substack{r,s=n+1 \\ s < r}}^{m-1} \sum_{\substack{j,k=1 \\ j \neq k}}^n \{e_i(R_{sjk}^r) \mathbf{x} + R_{sjk}^r e_i\} \wedge e_1 \wedge \cdots \wedge \underbrace{e_s}_{k\text{-th}} \wedge \cdots \wedge \underbrace{e_r}_{j\text{-th}} \wedge \cdots \wedge e_n \\
 & + \sum_{\substack{r,s=n+1 \\ s < r}}^{m-1} \sum_{\substack{j,k,l \\ j,k,l \neq}}^n R_{sjk}^r \left\{ \sum_{h=1}^n \omega_{lh}(e_i) \mathbf{x} \wedge e_1 \wedge \cdots \wedge \underbrace{e_h}_{l\text{-th}} \wedge \cdots \wedge \underbrace{e_s}_{k\text{-th}} \wedge \cdots \wedge \underbrace{e_r}_{j\text{-th}} \wedge \cdots \wedge e_n \right. \\
 & \left. + \sum_{t=n+1}^{m-1} h_{il}^t \mathbf{x} \wedge e_1 \wedge \cdots \wedge \underbrace{e_t}_{l\text{-th}} \wedge \cdots \wedge \underbrace{e_s}_{k\text{-th}} \wedge \cdots \wedge \underbrace{e_r}_{j\text{-th}} \wedge \cdots \wedge e_n \right\} \\
 & - \sum_{r,s=n+1}^{m-1} \sum_{\substack{j,k,l=1 \\ j \neq k}}^n R_{sjk}^r h_{il}^s \mathbf{x} \wedge e_1 \wedge \cdots \wedge \underbrace{e_l}_{k\text{-th}} \wedge \cdots \wedge \underbrace{e_r}_{j\text{-th}} \wedge \cdots \wedge e_n \\
 & + \sum_{r,s,t=n+1}^{m-1} \sum_{\substack{j,k=1 \\ j \neq k}}^n R_{sjk}^r \omega_{st}(e_i) \mathbf{x} \wedge e_1 \wedge \cdots \wedge \underbrace{e_t}_{k\text{-th}} \wedge \cdots \wedge \underbrace{e_r}_{j\text{-th}} \wedge \cdots \wedge e_n.
 \end{aligned} \tag{18}$$

From (14), (15) and (18) we know that $\|\hat{h}\|$ and scalar curvature are constant. Also, we have

$$\begin{aligned}
 \|\hat{h}\|^2 & \sum_{r=n+1}^{m-1} \sum_{k=1}^n h_{ik}^r \mathbf{x} \wedge e_1 \wedge \cdots \wedge e_{k-1} \wedge e_r \wedge e_{k+1} \wedge \cdots \wedge e_n - n \sum_{k=1}^n \delta_{ik} \hat{H} \wedge e_1 \wedge \cdots \wedge e_{k-1} \wedge \mathbf{x} \wedge e_{k+1} \wedge \cdots \wedge e_n \\
 & - \sum_{r,s=n+1}^{m-1} \sum_{\substack{j,k,l=1 \\ j \neq k}}^n R_{sjk}^r h_{il}^s \mathbf{x} \wedge e_1 \wedge \cdots \wedge \underbrace{e_l}_{k\text{-th}} \wedge \cdots \wedge \underbrace{e_r}_{j\text{-th}} \wedge \cdots \wedge e_n \\
 & = \lambda \sum_{r=n+1}^{m-1} \sum_{k=1}^n h_{ik}^r \mathbf{x} \wedge e_1 \wedge \cdots \wedge e_{k-1} \wedge e_r \wedge e_{k+1} \wedge \cdots \wedge e_n
 \end{aligned} \tag{19}$$

and

$$\begin{aligned}
 & n \sum_{k=1}^n \sum_{r=n+1}^{m-1} h_{ik}^r \hat{H} \wedge e_1 \wedge \cdots \wedge e_{k-1} \wedge e_r \wedge e_{k+1} \wedge \cdots \wedge e_n + \sum_{\substack{r,s=n+1 \\ s < r}}^{m-1} \sum_{\substack{j,k=1 \\ j \neq k}}^n R_{sjk}^r e_i \wedge e_1 \wedge \cdots \wedge \underbrace{e_s}_{k\text{-th}} \wedge \cdots \wedge \underbrace{e_r}_{j\text{-th}} \wedge \cdots \wedge e_n \\
 & = 0.
 \end{aligned} \tag{20}$$

Put $\hat{H} = \hat{\alpha} e_{n+1}$. It follows from (20) that $R_{sjk}^r = 0$ for $r, s \geq n + 2$ and $j, k = 1, \dots, n$. Also, we find $R_{sjk}^{n+1} = 0$ from $DH = 0$. Thus, the normal connection of M^n in \mathbb{S}^{m-1} is flat. Therefore, (20) yields

$$n\hat{\alpha} \sum_{k=1}^n \sum_{r=n+1}^{m-1} h_{ik}^r e_{n+1} \wedge e_1 \wedge \cdots \wedge e_{k-1} \wedge e_r \wedge e_{k+1} \wedge \cdots \wedge e_n = 0. \tag{21}$$

We see from (21) that the first normal space $\text{Im } h$ is spanned by e_{n+1} . Therefore, by the reduction theorem of Erbacher, we conclude that M^n is contained in a totally geodesic sphere $\mathbb{S}^{n+1} \subset \mathbb{S}^{m-1}$. Also, considering $h_{jk}^r = 0$ for $r = n + 2, \dots, m - 1$ and $j, k = 1, \dots, n$, and $R^D = 0$, we have from (19) that

$$n\hat{\alpha} \delta_{ik} + (\|\hat{h}\|^2 - \lambda) h_{ik}^{n+1} = 0 \tag{22}$$

for $i, k = 1, \dots, n$. If $\lambda = \|\hat{h}\|^2$, then (22) gives $\hat{\alpha} = 0$ which is a contradiction. So $\lambda \neq \|\hat{h}\|^2$ and by taking the sum of (22) for $i = k$ and i from 1 up to n we get $n\hat{\alpha}(n + \|\hat{h}\|^2 - \lambda) = 0$ which gives $\lambda = n + \|\hat{h}\|^2$, and thus $h_{ii}^{n+1} = \hat{\alpha} \neq 0$ for $i = 1, \dots, n$ from (22). Therefore, M is a non-totally geodesic and totally umbilical hypersurface of \mathbb{S}^{n+1} , and consequently M is an open part of a small n -sphere of \mathbb{S}^{n+1} which comes from the equation of Gauss.

Conversely, let M be an open part of a small n -sphere of a totally geodesic $(n + 1)$ -sphere $\mathbb{S}^{n+1} \subset \mathbb{S}^{m-1}$. Without loss of generality, we assume that M is immersed in $\mathbb{S}^{n+1} \subset \mathbb{E}^{n+2}$, that is, M is an open part of a small sphere $\mathbb{S}^n(x_0, c_0)$ of $\mathbb{S}^{n+1} \subset \mathbb{E}^{n+2}$ with the center $x_0 \in \mathbb{E}^{n+2}$ and curvature c_0 . Since M is a hypersurface of \mathbb{S}^{n+1} , the normal bundle of M in \mathbb{E}^{n+2} is flat.

Let $e_1, \dots, e_{n+1}, e_{n+2}$ be a local orthonormal frame field on M in \mathbb{E}^{n+2} such that e_1, \dots, e_n are tangent to M and $e_{n+1}, e_{n+2} = \mathbf{x}$ are normal to M . It is easy to show that the mean curvature $\hat{\alpha}$ of the small sphere $\mathbb{S}^n(x_0, c_0)$ is $\hat{\alpha} = |x_0|/\sqrt{1 - |x_0|^2}$, and $c_0 = 1 + \hat{\alpha}^2$ from the equation of Gauss. Also, the mean curvature vector $\hat{H} = \hat{\alpha}e_{n+1}$ is parallel in \mathbb{E}^{n+2} . Hence, from (8) we have

$$\Delta \tilde{\nu} = n\hat{\alpha}^2 \tilde{\nu} + n\hat{\alpha}e_{n+1} \wedge e_1 \wedge e_2 \wedge \dots \wedge e_n. \tag{23}$$

Now, if we put

$$c = \frac{1}{1 + \hat{\alpha}^2} (\tilde{\nu} - \hat{\alpha}e_{n+1} \wedge e_1 \wedge e_2 \wedge \dots \wedge e_n), \quad \tilde{\nu}_p = \frac{\hat{\alpha}}{1 + \hat{\alpha}^2} (\hat{\alpha}\tilde{\nu} + e_{n+1} \wedge e_1 \wedge e_2 \wedge \dots \wedge e_n),$$

we then have $\tilde{\nu} = c + \tilde{\nu}_p$. As $\hat{\alpha}$ is constant it is easily seen that $e_i(c) = 0, i = 1, \dots, n$, i.e., c is a constant vector. Using (9) and (23), we obtain from a direct computation that $\Delta \tilde{\nu}_p = n(\hat{\alpha}^2 + 1)\tilde{\nu}_p$. Therefore, the spherical Gauss map $\tilde{\nu}$ is non-mass-symmetric 1-type. □

Theorem 3.6. *Let M be a non-totally umbilical hypersurface in \mathbb{S}^{n+1} with nonzero constant mean curvature in \mathbb{S}^{n+1} . Then the spherical Gauss map $\tilde{\nu}$ is mass-symmetric and of 2-type if and only if M has constant scalar curvature.*

Proof. Let M be a non-totally umbilical hypersurface in $\mathbb{S}^{n+1} \subset \mathbb{E}^{n+2}$ with nonzero constant mean curvature $\hat{\alpha}$ in \mathbb{S}^{n+1} . Suppose that M has constant scalar curvature S . We show that the spherical Gauss map $\tilde{\nu}$ is mass-symmetric and of 2-type.

Let \mathbf{x} be the position vector of M in \mathbb{E}^{n+2} . Let $e_1, \dots, e_{n+1}, e_{n+2} = \mathbf{x}$ be a local orthonormal frame field on M in \mathbb{E}^{n+2} such that e_1, \dots, e_n are tangent to M and e_{n+1}, e_{n+2} normal to M which are parallel in the normal bundle of M in \mathbb{E}^{n+2} . Since M is a hypersurface of $\mathbb{S}^{n+1} \subset \mathbb{E}^{n+2}$, the normal bundle of M in \mathbb{E}^{n+2} is flat. We choose e_1, \dots, e_n such that $A_{e_{n+1}}(e_i) = h_{ii}^{n+1}e_i, i = 1, \dots, n$. As $h_{ij}^{n+1} = 0$ for $i \neq j$, it is easily seen that

$$n(\|\hat{h}\|^2 - n\hat{\alpha}^2) = \sum_{i < j} (h_{ii}^{n+1} - h_{jj}^{n+1})^2 \geq 0, \tag{24}$$

and equality holds if and only if M^n is totally umbilical. If we put $D_0 = (\|\hat{h}\|^2 - n)^2 + 4n^2\hat{\alpha}^2 > 0$, then

$$\|\hat{h}\|^2 + n - \sqrt{D_0} = \frac{(\|\hat{h}\|^2 + n)^2 - D_0}{\|\hat{h}\|^2 + n + \sqrt{D_0}} = \frac{4n(\|\hat{h}\|^2 - n\hat{\alpha}^2)}{\|\hat{h}\|^2 + n + \sqrt{D_0}} > 0 \tag{25}$$

as M is non-totally umbilical. Since the mean curvature $\hat{\alpha}$ and the scalar curvature S are constants, equation (6) implies that $\|\hat{h}\|^2$ is constant. Hence D_0 is constant.

Now, as the normal bundle is flat, i.e., $R^D = 0$ and $\hat{H} = \hat{\alpha}e_{n+1}$ is parallel, equation (8) becomes

$$\Delta \tilde{\nu} = \|\hat{h}\|^2 \tilde{\nu} + n\hat{\alpha}e_{n+1} \wedge e_1 \wedge e_2 \wedge \dots \wedge e_n. \tag{26}$$

If we put

$$\begin{aligned} \tilde{\nu}_p &= -\left(\frac{\|\hat{h}\|^2 - n - \sqrt{D_0}}{2\sqrt{D_0}}\right)\tilde{\nu} - \frac{n\hat{\alpha}}{\sqrt{D_0}}e_{n+1} \wedge e_1 \wedge e_2 \wedge \dots \wedge e_n, \\ \tilde{\nu}_q &= \left(\frac{\|\hat{h}\|^2 - n + \sqrt{D_0}}{2\sqrt{D_0}}\right)\tilde{\nu} + \frac{n\hat{\alpha}}{\sqrt{D_0}}e_{n+1} \wedge e_1 \wedge e_2 \wedge \dots \wedge e_n, \end{aligned}$$

then we have $\tilde{\nu} = \tilde{\nu}_p + \tilde{\nu}_q$, and by using (9) and (26) a direct computation shows that

$$\Delta \tilde{\nu}_p = \lambda_q \tilde{\nu}_p \quad \text{and} \quad \Delta \tilde{\nu}_q = \lambda_q \tilde{\nu}_q,$$

where $\lambda_p = \frac{1}{2}(\|\hat{h}\|^2 + n - \sqrt{D_0}) > 0$ because of (25) and $\lambda_q = \frac{1}{2}(\|\hat{h}\|^2 + n + \sqrt{D_0}) > 0$ which are constants. Therefore, the spherical Gauss map \tilde{v} is mass-symmetric and of 2-type.

Conversely, suppose that the spherical Gauss map \tilde{v} is mass-symmetric and of 2-type. Then \tilde{v} admits a spectral decomposition of the form

$$\tilde{v} = \tilde{v}_p + \tilde{v}_q, \quad \Delta\tilde{v}_p = \lambda_q\tilde{v}_p, \quad \Delta\tilde{v}_q = \lambda_p\tilde{v}_q \tag{27}$$

with $\lambda_p < \lambda_q$, where \tilde{v}_p and \tilde{v}_q are non-constant maps. Then, we find from (27) that

$$\Delta^2\tilde{v} = (\lambda_p + \lambda_q)\Delta\tilde{v} - \lambda_p\lambda_q\tilde{v}. \tag{28}$$

Since the normal bundle is flat and $\hat{H} = \hat{\alpha}e_{n+1}$ is parallel, we have (26). By using (9) and (26), we obtain that

$$\Delta^2\tilde{v} = (n + \|\hat{h}\|^2)\Delta\tilde{v} + (\Delta\|\hat{h}\|^2 + n^2\hat{\alpha}^2 - n\|\hat{h}\|^2)\tilde{v} - 2 \sum_{i=1}^n h_{ii}^{n+1} e_i(\|\hat{h}\|^2) \mathbf{x} \wedge e_1 \wedge \dots \wedge e_{i-1} \wedge e_{n+1} \wedge e_{i+1} \wedge \dots \wedge e_n. \tag{29}$$

Comparing (28) and (29), the coefficient of $\Delta\tilde{v}$ implies that $n + \|\hat{h}\|^2 = \lambda_p + \lambda_q$, thus $\|\hat{h}\|^2$ is constant. Therefore the scalar curvature of M is constant by (6). □

As isoparametric hypersurfaces in \mathbb{S}^{n+1} have constant scalar curvature, we state the following corollary.

Corollary 3.7. *Every non-totally umbilical isoparametric hypersurface M in \mathbb{S}^{n+1} with nonzero mean curvature $\hat{\alpha}$ in \mathbb{S}^{n+1} has mass-symmetric 2-type spherical Gauss map.*

For example, the product submanifold $\mathbb{S}^k(a) \times \mathbb{S}^{n-k}(b) \subset \mathbb{S}^{n+1}$ with $a^2 + b^2 = 1$ and $a \neq b$ is a non-totally umbilical isoparametric hypersurface of \mathbb{S}^{n+1} which has mass-symmetric and 2-type spherical Gauss map.

Theorem 3.8. *A non-totally umbilical surface M of \mathbb{S}^3 with nonzero constant mean curvature in \mathbb{S}^3 has the mass-symmetric 2-type spherical Gauss map \tilde{v} if and only if M is an open part of $\mathbb{S}^1(a) \times \mathbb{S}^1(b) \subset \mathbb{S}^3$, where $a \neq b$ and $a^2 + b^2 = 1$.*

Proof. First we assume that M is an open part of $\mathbb{S}^1(a) \times \mathbb{S}^1(b)$ in $\mathbb{S}^3(1) \subset \mathbb{E}^4$ which is defined by

$$\mathbf{x}(u, v) = \left(a \cos \frac{u}{a}, a \sin \frac{u}{a}, b \cos \frac{v}{b}, b \sin \frac{v}{b} \right),$$

where $a \neq b$ and $a^2 + b^2 = 1$. It is well known that M is a non-totally umbilical isoparametric surface. Also, it is not minimal as $a \neq b$. By Corollary 3.7, M has the mass-symmetric 2-type spherical Gauss map.

For later use we need the connection forms of M . We choose

$$e_1 = \frac{\partial}{\partial u}, \quad e_2 = \frac{\partial}{\partial v}, \quad e_3 = \left(b \cos \frac{u}{a}, b \sin \frac{u}{a}, -a \cos \frac{v}{b}, -a \sin \frac{v}{b} \right), \quad e_4 = \mathbf{x}$$

which form an orthonormal frame field on M . By a direct computation we have

$$\omega_1 = du, \quad \omega_2 = dv, \quad \omega_{12} = \omega_{34} = 0, \quad \omega_{13} = -\mu_0\omega_1, \quad \omega_{23} = \frac{1}{\mu_0}\omega_2, \quad \omega_{14} = -\omega_1, \quad \omega_{24} = -\omega_2, \tag{30}$$

where $\mu_0 = b/a$.

Conversely, suppose that M is a non-totally umbilical surface of \mathbb{S}^3 with nonzero constant mean curvature $\hat{\alpha}$, and the spherical Gauss map \tilde{v} is mass-symmetric and of 2-type. Then, by Theorem 3.6, M has constant scalar curvature S , that is, from (6) the scalar curvature S of M is $S = 2 + 4\hat{\alpha}^2 - \|\hat{h}\|^2$ which is constant. If we choose an orthonormal tangent frame on M such that $A_3(e_i) = h_{ii}^3 e_i$, $i = 1, 2$, then the constancy of S and $\hat{\alpha}$ imply that the principal curvatures h_{11}^3 and h_{22}^3 of A_3 are constants.

Now, considering the Codazzi equation (3) we have $(h_{ii}^3 - h_{jj}^3)\omega_{ij}(e_j) = 0$, $i \neq j$ that gives $\omega_{ij}(e_j) = 0$, $j = 1, 2$, as M is non-totally umbilical. So, M is flat, and from the equation of Gauss we have $h_{11}^3 h_{22}^3 = -1$. If we put $\mu_0 = -h_{11}^3$, then $h_{22}^3 = 1/\mu_0$.

Since M is flat, we can choose a local coordinate (u, v) on M with $\omega_1 = du$, $\omega_2 = dv$. So, we have

$$\omega_{12} = \omega_{34} = 0, \quad \omega_{13} = -\mu_0\omega_1, \quad \omega_{23} = \frac{1}{\mu_0}\omega_2, \quad \omega_{14} = -\omega_1, \quad \omega_{24} = -\omega_2. \tag{31}$$

Thus the connection forms ω_{AB} of M coincide with the connection forms of $\mathbb{S}^1(a) \times \mathbb{S}^1(b)$, $a \neq b$, given in (30). As a result of the fundamental theorem of submanifolds, M is locally isometric to $\mathbb{S}^1(a) \times \mathbb{S}^1(b)$. □

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